APPROXIMATION AND BLOW-UP PROBLEMS IN STOCHASTIC DIFFERENTIAL EQUATIONS

by

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ABSTRACT

In the first part of this work, we will show weak convergence of probability measures. The measure corresponding to the solution of the following one-dimensional nonlinear stochastic heat equation $\frac{\partial}{\partial t}u_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\eta_\alpha$ with colored noise η_α will converge to the measure corresponding to the solution of the same equation but with white noise η, as $\alpha \uparrow 1$. Function σ is taken to be Lipschitz and the Gaussian noise η_α is assumed to be colored in space and its covariance is given by $\mathrm{E}\left[\eta_\alpha(t,x)\eta_\alpha(s,y)\right] = \delta(t-s)f_\alpha(x-y)$ where f_α is the Riesz kernel $f_\alpha(x) \propto 1/\left|x\right|^\alpha$. We will work with the classical notion of weak convergence of measures, that is convergence of probability measures on a space of continuous functions with compact domain and sup-norm topology. We will also state a result about continuity of measures in α, for $\alpha \in (0,1)$.

In the second part of this work, we will show existence and blow-up of the solution to $\frac{\partial}{\partial t}u_t(x) = \mathcal{L}u_t(x) + \sigma(u_t(x))\eta$ on a circle with white noise η. The operator $\mathcal{L}$ is taken to be the generator of a Lévy process and σ is a nonlinear function of form $\sigma(x) \propto |x|^\gamma$, for $\gamma > 1$. We will develop precise condition for existence or blow-up of the solution in terms of γ and the Lévy process corresponding to $\mathcal{L}$.

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NOTATION AND SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of natural numbers
$\mathscr{S}$	Space of Schwartz functions that are infinitely differentiable and rapidly decreasing
$\mathscr{S}^*$	Space of generalized functions, i.e., continuous linear functionals on $\mathscr{S}$
$\mathscr{B}(\mathbb{R})$	Borel σ-algebra on $\mathbb{R}$. That is, σ-algebra generated by Borel sets
$\mathcal{C}$	Space of continuous functions
$\mathcal{C}_b$	Space of bounded continuous functions
$\mathcal{C}_0^k$	Space of k times continuously differentiable functions vanishing at infinity
a.s.	Almost surely
$\mathcal{F}$	Fourier transform
$\hat{f}$	Fourier transform of a function f
$f * g(x)$	Convolution of two function, $f * g(x) := \int f(y)g(x-y)dy$
Ω	Probability space
$\mathscr{F}, \mathscr{G}$	Sigma algebras on Ω
$\mathscr{F} \vee \mathscr{G}$	Smallest sigma algebra containing $\mathscr{F}$ and $\mathscr{G}$
$\mathscr{F}_\infty$	Smallest sigma algebra containing $\mathscr{F}_t$ for every nonnegative t, $\mathscr{F}_\infty := \bigvee_{t \geq 0} \mathscr{F}_t$
$\mathscr{F}_\tau$	Stopped sigma algebra, $\mathscr{F}_\tau := \{A \in \mathscr{F}_\infty : A \cap \{\tau \leq t\} \in \mathscr{F}_t\}$
$\mathcal{N}_{\gamma,k}(u)$	$\mathcal{N}_{\gamma,k}(u) = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t} \left\Vert u_t \right\Vert_{L^k(P)}\right) \quad , \quad k \geq 2.$
$\vee$	Maximum of two number $a \vee b = \max(a,b)$
$\wedge$	Minimum of two number $a \wedge b = \min(a,b)$
$\langle M \rangle$	Quadratic variation process of a continuous martingale M
$\mathbb{1}_A$	Indicator function of a set A
$\lfloor \cdot \rfloor$	Floor function, $\lfloor a \rfloor := \sup\{j \in \mathbb{Z} : j \leq a\}$
$\lceil \cdot \rceil$	Ceil function, $\lceil a \rceil := \sup\{j \in \mathbb{Z} : j < a+1\}$
$\Vert \cdot \Vert_{L^k(P)}$	L^k norm on probability space, $\Vert X \Vert_{L^k(P)} := \mathrm{E}\left[\lvert X \rvert^k\right]^{1/k}$

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CHAPTER 1

INTRODUCTION

Partial differential equations describe many important physical phenomena, such as dissipation of heat, diffusion, propagation of waves or behavior of fluids. The Schrödinger equation in quantum physics describes time evolution of wave functions which give us probability distributions for particles at given time. This work will focus on a single class of equations. We will consider heat equations with noise, also known as the Stochastic Heat Equations. In the simplest case, the noise entering the heat equation is additive

$$\partial_t u = \Delta u + \text{“}noise\text{”},$$

and sometimes we consider heat equations with multiplicative noise of the form

$$\partial_t u = \Delta u + \sigma(u) \cdot \text{“}noise\text{”}.$$

The Stochastic Heat Equation is simply a heat equation with random forcing. The above equations not only describe real-world phenomena, but they are also worth studying for their mathematical beauty. An equation of type

$$\partial_t u = \Delta u + u \cdot \text{“}noise\text{”}.$$

is a continuous limit of the so-called *parabolic Anderson*[1] model. Anderson was studying in [1] propagation of wave functions in random environment.

The Stochastic Heat Equation has deep connections to other partial differential equations with random forcing. The KPZ equation [35] describes growth of random media and is formally defined as:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \text{“}noise\text{”}.$$

The solution h of the above equation can be obtained from the parabolic Anderson model through the Hopf-Cole transformation

$$h = \log(u).$$

[1]Philip W. Anderson received the Nobel prize in Physics in 1977

There are numerous articles studying the KPZ equation. We will only mention Hairer's[2] seminal work [31].

Another equation that has connections to the Stochastic Heat Equation is the Stochastic Burgers' equation. That is (see [7, 49, 32]) the classical Burgers' equation with random forcing, that is

$$\partial_t v = \partial_x^2 v - 2v\partial_x v + \text{“}\textit{noise}\text{”}.$$

The above equation can be obtained from the Stochastic Heat Equation by the Hopf-Cole transformation $v = \partial_x \log(u)$ [7] of the solution to the parabolic Anderson model. It can also be thought of as the derivative of the solution to the KPZ equation. Overall, the above introduced equations "*formally*" satisfy the following:

$$\text{Parablic Anderson} \xrightarrow{\log} \text{KPZ} \xrightarrow{-\partial_x} \text{Burgers' equation} .$$

1.1 Meaning of the main theorems

We will present three main theorems throughout this work, one in each chapter. In this section, we present the three main theorems of this thesis. We also describe how the theorems relate to one another, as well as to the theory of Stochastic Partial Differential Equations. Each theorem is proved subsequently in a separate chapter.

1.1.1 Weak convergence

The first main theorem states that the solution to the nonlinear Stochastic Heat Equation

$$\begin{aligned} \frac{\partial}{\partial t} u &= \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x))\eta(t,x) \\ u(0,x) &= w(x), \end{aligned}$$

with *white noise* η can be approximated by the solution to the same equation, but with a noise that has spatial correlations. This type of noise is often called *colored* in space. The idea of approximation equation with white noise by equation with colored noise is not new. The authors in [6] showed that this kind of approximation can be done if we smooth the noise in the spatial variable by infinitely differential function with compact support. Our approximation will be a lot more singular; we will show that this approximation is

[2] Martin Hairer received the Fields medal in 2014 for his contributions to the theory of stochastic partial differential equations

possible if the *Fourier transform* of spatial correlations converge to Fourier transform of spatial correlation of white noise. Also, the spatial correlations of the colored noise will decay very slowly. The correlation between two points x and y will be proportional to

$$|x-y|^{-\alpha} \ , \ \alpha \in (0,1).$$

The authors in [3] independently showed that this type of approximation converges in the space of square integrable random variables by using the Malliavin Calculus techniques [45]. In Chapter 2, we will show that the convergence holds in a much stronger sense.

Sometimes it is better to consider the Stochastic Heat Equation with colored noise and think of it as an approximation to the Stochastic Heat Equation with white noise. For example, in [spatial] dimension 2 and higher [20, Ex. 3.1], it is not known how to define a solution! There has been some attempts to overcome this issue. Authors in [33] allowed for time correlations of the noise. Another attempt in [44] considered the same type of spatial covariance structure but their result holds for spatial dimensions higher or equal to 3.

1.1.2 Existence

In the Existence chapter, we consider a more general equation on a circle

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \mathcal{L}u(t,x) + \sigma(u(t,x))\eta(t,x) \\ u(0,x) &= w(x), \ x \in [0,2\pi], \end{aligned} \tag{1.1}$$

with a linear operator $\mathcal{L}$. We take $\mathcal{L}$ to be the generator of Lévy process on a circle. For the moment, we may think of $\mathcal{L}$ as a fractional derivative or a fractional Laplacian. There are numerous articles studying equations of type (1.1); see for example [13]. We ask ourselves how fast can σ grow if we do not want the solution u to '*blow-up*' in finite time. First, let us look at results for Ordinary and Partial Differential Equations.

For Ordinary Differential Equations of type

$$\begin{aligned} y' &= f(y) \\ y(0) &= c, \end{aligned} \tag{1.2}$$

with positive and continuous function f, the finite-time blow-up occurs [27, 47] if and only if

$$\int_1^\infty f(x)^{-1}dx < \infty.$$

There are similar results in the theory of Partial Differential Equations. The solution to

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u^\gamma(t,x) \tag{1.3}$$

$$u(0,x) = w(x) \tag{1.4}$$

blows-up [26] in the supremum norm in finite time if $1 < \gamma < 3$ for every positive initial condition $w(x)$. If $\gamma > 3$, then there exist initial conditions that give nonexplosive solutions. Similar results hold if we consider (1.3) defined on a bounded domain [5, Thm. 3.2]. Blow-up problems for partial differential equations are nicely summarized in [27].

It appears that the forcing term cannot be of power greater than 1. For $f(x) = |x|^{1+\epsilon}$, we get blow-up in (1.2). Similarly, we get blow-up in (1.3) if we force the equation with $\gamma > 1$. This is no longer true if we add randomness to our equations. For the stochastic ordinary differential equations, we have Feller's criterion for explosion [23],[34, Thm. 5.5.29]. Stochastic ordinary differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

with $\sigma^2(x) > 0$, has a positive probability to blow-up in finite time if

$$\min(v(\infty), v(-\infty)) < \infty,$$

where

$$v(x) = \int_c^x p'(y)\int_c^y \frac{2dz}{p'(z)\sigma^2(z)}dy \quad \text{and} \quad p'(x) = \exp\left(-2\int_c^x \frac{b(\xi)d\xi}{\sigma^2(\xi)}\right).$$

As we can see, the condition on blow-up is a lot more interesting. For example,

$$dX_t = |X_t|^\gamma \, dB_t$$

never blows-up for any $\gamma > 1$!

Explosion criterion is no less interesting if we talk about the Stochastic Heat Equation on a circle, that is equation (1.1) with $\mathcal{L} = \partial^2/\partial x^2$. The author in [15] gave a criterion for explosion of $L^p(\Omega \times \mathbb{R})$ norm of a solution. More interestingly, the author in [40] showed that for $\mathcal{L} = \partial^2/\partial x^2$ and $\sigma(x) = |x|^\gamma, 1 \le \gamma < 3/2$, the solution *exists for all times* on a bounded domain! Moreover, we have the converse result [42] which tells us that the solution blows-up in finite time with positive probability if $\gamma > 3/2$.

We asked ourselves, why is the critical exponent $3/2$? Does it change if $\mathcal{L}$ is a generator of a Lévy process or fractional derivative? We answer this question for a large class of Lévy processes and fractional derivatives. A precise answer can be found in Chapter 3, Theorem 3.1.1. We found that the critical exponent for which solution exists is largest when $\mathcal{L} = \partial^2/\partial x^2$.

1.1.3 Blow-up

A question that comes naturally is: *"When does the solution blow-up?"* The authors of reference [43] considered (1.1) in the case that $\mathcal{L} = \partial^2/\partial x^2$ and $\sigma(x) = |x|^\gamma$, and showed that the solution has positive probability of finite-time blow-up when γ is sufficiently large. Mueller in [42] shows that the solution explodes for γ larger than $3/2$. Nothing is known about the case when $\gamma = 3/2$.

In Chapter 4, we show blow-up. We considered a smaller class of operators $\mathcal{L}$, since the proof relies heavily on scaling. For some operators, we were able to get one exponent γ_1, such that we have explosion if $\gamma > \gamma_1$ and existence of a solution for all times if $\gamma < \gamma_1$. There are some operators $\mathcal{L}$ for which we were not able to find one such γ_1. We rather found γ_1 and γ_2 such that for $\gamma > \gamma_2$, we have explosion and existence for $\gamma < \gamma_1$. To our surprise, the point where we start to have two coefficients γ_1, γ_2 is characterized by the *golden ratio*.

1.2 Probability preliminaries

This section will cover basic definitions that will be used throughout this work. We will work on an abstract probability space which is a triplet $(\Omega, \mathscr{F}, \mathrm{P})$. The first element Ω is a set, $\mathscr{F}$ denotes a *sigma algebra* of subsets of Ω and P is a probability measure. Let us briefly mention that for $\mathscr{F}$ to be a sigma algebra, it must have the following properties:

1. $\emptyset \in \mathscr{F}$;

2. If $A \in \mathscr{F}$, then $\Omega \setminus A \in \mathscr{F}$;

3. If $A_i \in \mathscr{F}$ for $i \in \mathbb{N}$, then $\cup_{i\in\mathbb{N}} A_i \in \mathscr{F}$.

A probability measure P is a measure (that is, a set function from $\mathscr{F}$ to $[0, +\infty]$) such that $\mathrm{P}(\Omega) = 1$. A *random variable* is a measurable map from Ω to a metric space E [51, p. 10]. Or course, we need to specify what kind of sigma algebra on E we have in mind. We are going to take $\mathscr{B}(E)$, the Borel sigma algebra on E. Similarly, *stochastic process* is a collection $\{X_i\}_{i\in I}$ of random variables. Typically, the index set I is $[0, \infty)$ or $[0, \infty) \times \mathbb{R}$. Unless specified otherwise, we will consider $I = [0, \infty)$ in the rest of this section.

Before we introduce *Martingales*, we should say something about filtration. A *filtration* [51, Def. 1.4.1] is an increasing collection of sigma algebras $\mathscr{F}_t, t \in [0, \infty)$ such that $\mathscr{F}_t \subset \mathscr{F}$. By increasing, we precisely mean that $\mathscr{F}_s \subset \mathscr{F}_t$ for $s < t$. Let us define $\mathscr{F}_\infty$ to be the smallest sigma algebra that contains $\mathscr{F}_t$ for every $t \in [0, \infty)$. We often write

$$\mathscr{F}_\infty := \bigvee_{t\geq 0} \mathscr{F}_t = \sigma\left(\bigcup_{t\geq 0} \mathscr{F}_t\right).$$

In general, there is no guarantee that union of two sigma algebras is a sigma algebra, as the following example demonstrates.

Example 1.2.1. *Suppose that* $\Omega = \{1,2,3,4,5,6\}$ *and we have two sigma algebras* $\mathscr{F} = \{\{\},\{1,2,3,4,5,6\},\{1,2,3\},\{4,5,6\}\}$ *and* $\mathscr{G} = \{\{\},\{1,2,3,4,5,6\},\{3,4\},\{1,2,5,6\}\}$*. Union of* $\mathscr{F}$ *and* $\mathscr{G}$ *is not a sigma algebra.*

Proof. We have

$$\mathscr{F} \cup \mathscr{G} = \mathscr{F} = \{\{\},\{1,2,3,4,5,6\},\{1,2,3\},\{4,5,6\},\{3,4\},\{1,2,5,6\}\}.$$

The above set is definitely not a sigma algebra, since it does not contain $\{3\}$ which can be formed as $\{1,2,3\} \cap \{3,4\}$. $\square$

We will sometimes need the intersection of sigma algebras. It is true that intersection of sigma algebras is a sigma algebra. We will write $\bigwedge_{t\in I} \mathscr{F}_t$ for the intersection of sigma algebras in some index set I. This notation is adopted from [51].

We will say that a random variable is *integrable* if

$$\mathrm{E}[|X|] := \int_\Omega |X| \, d\,\mathrm{P} < \infty,$$

assuming that the above quantity makes sense. The *expectation*, denoted by E, is defined by the above relation. In other words, it is simply an integral with respect to the probability measure P. We will also need to define [abstract] *conditional expectation*. We will not discuss very deeply the meaning of conditional expectation. We can refer the reader for example to [46, Appendix B] and [16, Ch. 9.1]. Suppose that $\mathscr{G} \subset \mathscr{F}$, then the *conditional expectation* of (integrable) random variable X, given sigma algebra $\mathscr{G}$, denoted by

$$\mathrm{E}[X|\mathscr{G}]$$

is a $\mathscr{G}$-measure random variable $\mathrm{E}[X|\mathscr{G}]$ such that for every $A \in \mathscr{G}$, we have

$$\int_\Omega \mathbb{1}_A \cdot \mathrm{E}[X|\mathscr{G}] \, d\,\mathrm{P} = \int_\Omega \mathbb{1}_A \cdot X d\,\mathrm{P}.$$

Conditional expectations are almost surely unique. That is, if there exists another $\mathscr{G}$-measurable random variable Y that satisfies

$$\int_\Omega \mathbb{1}_A \cdot Y d\,\mathrm{P} = \int_\Omega \mathbb{1}_A \cdot X d\,\mathrm{P}$$

for every $A \in \mathscr{G}$, then $Y = \mathrm{E}[X|\mathscr{G}]$ almost surely.

Definition 1.2.2 (Martingale). *[51, II.1.1] A real-valued process $X_t, t \geq 0$ is a $\mathscr{F}_t$ martingale if*

1. $\mathrm{E}[|X_t|] < \infty$ *for every* $t \geq 0$*;*
2. $\mathrm{E}[X_t|\mathscr{F}_s] = X_s$ *almost surely for every* s, t *such that* $s < t$*;*
3. X_t *is adapted to filtration* $\mathscr{F}_t$*, that is* X_t *is* $\mathscr{F}_t$ *measurable for every* $t \geq 0$*.*

Even though martingales form a large class of stochastic processes, we will need to generalize them a little bit. Whenever the filtration of interest is determined by the context, we will simply write *'martingale'* instead of *'$\mathscr{F}_t$ martingale'*. The definition of a stopping time and local Martingale follows.

Definition 1.2.3 (Stopping time). *[51, I.4.4] A stopping time τ relative to the filtration $\mathscr{F}_t$ is a map on Ω with values in $[0, \infty]$, such that for every t,*

$$\{\tau \leq t\} \in \mathscr{F}_t.$$

Definition 1.2.4 (Continuous Local Martingale). *[51, IV.1.5] An adapted, continuous process X is an $\mathscr{F}_t$ local martingale if there exist stopping times $\tau_n, n \in \mathbb{N}$, such that*

1. *the sequence* τ_n *is increasing and* $\lim_{n\to\infty} \tau_n = +\infty$ *almost surely;*
2. *for every* n*, the process* $X_{\tau_n \wedge t}$ *is* $\mathscr{F}_t$ *martingale.*

We will also need to work with semimartingales. The definition follows.

Definition 1.2.5 (Continuous Semimartingale). *[51, Ch IV.1.17] A continuous $\mathscr{F}_t$ semimartingale is a continuous process X which can be written as $X = M + A$, where M is a continuous $\mathscr{F}_t$ local martingale and A is a continuous adapted process of finite variation (a.s. finite variation).*

The following line illustrates the relationship between martingales, local martingales and semimartingales:

$$\text{Continous martingale} \subset \text{Continous local martingale} \subset \text{Continous semimartingale.}$$

The inclusion in the above line is strict. That is, there are continuous local martingales which are not martingales. Under some circumstances, continous local martingales are martingales. If a local martingale is of class DL, then it is a martingale [51, IV.1.7]. The definition of class DL follows.

Definition 1.2.6 (Class DL)**.** *[51, Ch. IV.1.6] A real valued adapted process X is said to be of class DL if for every $a > 0$, the family of random variables X_T, where T ranges through all stopping times less than a, is uniformly integrable.*

Definition 1.2.7 (Uniformly integrable)**.** *[16, p. 100] A family of random variable $X_t, t \in I$ is said to be uniformly integrable if*

$$\lim_{N \to \infty} \int_{\Omega} \mathbb{1}_{|X_t| > A} \cdot |X_t| \, d\,\mathrm{P} = 0,$$

uniformly in $t \in I$.

Definition 1.2.8 (Quadratic Variation)**.** *[51, IV.1.8] If M is a continous local martingale, there exists a unique increasing continuous process $\langle M, M \rangle$ vanishing at zero, such that $M^2 - \langle M, M \rangle$ is a continuous local martingale. Process $\langle M, M \rangle$ is called the quadratic variation of M. Moreover,*

$$\sup_{0 \leq s \leq t} \left(\langle M, M \rangle_s - \sum_{i=0}^{\infty} \left(M_{s \wedge t_{i+1}} - M_{s \wedge t_i} \right)^2 \right)$$

converges to zero in probability as $\sup_{i \in \mathbb{N}} |t_i - t_{i-1}| \to 0$, where $\{t_0 = 0, t_1, \dots\}$ is a partition of an interval $[0, \infty)$.

For the most part, we will work only with continuous processes. There is going to be one important exception to that rule: Lévy processes.

1.3 Lévy processes

We will define Lévy processes on the real line as well as some preliminaries that will be used in Chapters 3 and 4. A real valued stochastic process X_t is a Lévy process if the following conditions are satisfied [59, Def. 1.6]:

1. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. $X_0 = 0$ almost surely.

3. The distribution of $X_{s+t} - X_s$ does not depend on s.

4. X_t is continuous in probability, that is

$$\lim_{s \to t} \mathrm{P}\left(|X_t - X_s| > \epsilon \right) = 0,$$

for every $\epsilon > 0$ and every $t \geq 0$.

5. X_t has almost surely right continuous paths with left limits. (We will sometimes say that X_t has càdlàg or RCLL paths.)

Lévy processes on the real line have a beautiful characterization, known as Lévy-Khintchine formula, which describes the Fourier transform of a measure (sometimes density) of X_t at any time $t \geq 0$.

Theorem 1.3.1 (Lévy-Khintchine formula)**.** *[59, Thm. 8.1 & Thm. 7.10] Let $\{X_t : t \geq 0\}$ be a Lévy process taking values in $\mathbb{R}^d$, then for all $t \geq 0, \xi \in \mathbb{R}^d$, we have*

$$\begin{aligned}\mathrm{E}\left[e^{i\xi X_t}\right] &= \mathrm{E}\left[e^{i\xi X_1}\right]^t \\ &= \exp\left(-t\left(\frac{1}{2}(\xi, A\xi) + i(\gamma, \xi) + \int_{\mathbb{R}^d}\left(1 - e^{i(z,\xi)} + i(z,\xi)\mathbb{1}_{|z|<1}(z)\right)m(dz)\right)\right),\end{aligned}$$

where $\gamma \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is nonnegative definite matrix and m is a measure on $\mathbb{R}^d$ that satisfies

$$m(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)m(dx) < \infty.$$

Lastly, we will need to define the generator of a Lévy process. The generator $\mathcal{L}$ of a Lévy process acting on function f will be

$$\mathcal{L}f(x) = \lim_{t\downarrow 0}\frac{\mathrm{E}\left[f(x + X_t)\right] - f(x)}{t}. \tag{1.5}$$

Naturally, one might ask for which function does the limit on the right-hand side exist? That is, what is the domain of the generator. The limit on the right-hand side exists for any $f \in \mathcal{C}_0^2(\mathbb{R})$ by [59, Thm. 31.5] or twice continuously differentiable functions with bounded derivatives. Space $C_0^k(\mathbb{R})$ consists of k times continuously differentiable function which vanish, together with their derivatives at infinity.

Example 1.3.2 (Generator of Brownian Motion)**.** *Suppose that $f \in \mathcal{C}^2(\mathbb{R})$ and B_t is the standard Brownian motion. Then,*

$$\mathcal{L}f(x) := \lim_{t\downarrow 0}\frac{\mathrm{E}\left[f(x + B_t)\right] - f(x)}{t} = \frac{1}{2}f''(x).$$

Proof. The form of generator follows almost directly from application of the Itô's Lemma. Write,

$$\mathrm{E}\left[f(x + B_t) - f(x)\right] = \mathrm{E}\left[\int_0^t f'(x + B_s)dB_s + \frac{1}{2}\int_0^t f''(s)ds\right].$$

The previous line immediately yields

$$\mathcal{L}f(x) = \lim_{t \downarrow 0} \frac{\mathrm{E}\left[\int_0^t \frac{1}{2} f''(x + B_s) ds\right]}{t} = \frac{1}{2} f''(x).$$

□

1.4 Stochastic integration and SPDEs

We will give a meaning to the following Stochastic Heat Equation

$$\begin{aligned} \frac{\partial}{\partial t} u(x) &= \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x))\eta(t,x) \\ u(0,x) &= w(x), \end{aligned} \tag{1.6}$$

where σ is Lipshitz continuous, the initial condition g is positive and bounded function and η denotes the white noise. First of all, let us define the *mild solution* for the above equation. Consider a partial differential equation of the form,

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(u(t,x)) \\ u(0,x) &= w(x). \end{aligned} \tag{1.7}$$

Definition 1.4.1 (Schwartz function)**.** *[28, p. 16] The space of Schwartz functions is a space of infinitely differentiable functions that decay rapidly together with their derivatives at infinity, that is*

$$\mathcal{S} = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} \left| x^k f^{(q)}(x) \right| < \infty \text{ for every } k, q \in \mathbb{N} \cup \{0\} \right\}.$$

We say that φ_n converges to φ in $\mathcal{S}$ if in every bounded region, the derivatives of all orders of φ_n converge uniformly to the corresponding derivatives of φ.

Definition 1.4.2 (Generalized function)**.** *The space of continuous linear functional from $\mathcal{S}$ to $\mathbb{R}$, denoted by*

$$\mathcal{S}',$$

is often called a space of generalized functions.

Let us assume for the moment that there exists a function that satisfies (1.3). Multiply equation (1.3) by a Schwartz function $\varphi \in \mathscr{S}(\mathbb{R}^2)$ and integrate with respect to time and space to obtain

$$\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial s} u(s,x)\varphi(s,x)dxds = \int_0^t \int_{\mathbb{R}} \frac{1}{2}\frac{\partial^2}{\partial x^2} u(s,x)\varphi(s,x)dxds + \int_0^t \int_{\mathbb{R}} \varphi(s,x)\sigma(u(s,x))dxds.$$

Use integration by parts[3] and get that

$$\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial s} u(s,x)\varphi(s,x)dxds = \int_0^t \int_{\mathbb{R}} u(s,x)\frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(s,x)dxds + \int_0^t \int_{\mathbb{R}} \varphi(s,x)\sigma(u(s,x))dxds.$$

The space $C_b^2(\mathbb{R})$ denotes the space of twice continuously differentiable function with bounded derivatives. Proceed with integration by parts in the time variable and write

$$\begin{aligned}&\int_{\mathbb{R}} \left(u(t,x)\varphi(t,x) - u(0,x)\varphi(0,x)\right) dx \\ &= \int_0^t \int_{\mathbb{R}} u(s,x)\left(\frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(s,x)\right) dxds + \int_0^t \int_{\mathbb{R}} \varphi(s,x)\sigma(u(s,x))dxds.\end{aligned}$$

If we choose[4]

$$\varphi(s,x) = p_{t-s}(x-y)$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right),$$

then we finally arrive at the following, so-called mild solution, to the classical heat equation:

$$u(t,y) = \int_{\mathbb{R}} u(x)p_t(y-x)dx + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,x))dxds. \tag{1.8}$$

Similarly to (1.8) we will define a *mild solution* to (1.6) as

$$u(t,x) = p_t * w(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))\eta(ds,dy). \tag{1.9}$$

At this point, it is not even clear what we mean by the stochastic integral in the equation (1.9). Loosely speaking, the integral on the right-hand side of (1.9) is an extension of the classical stochastic integral known from the theory of stochastic differential equation [51, 46, 34].

---

[3]Provided that u does not grow too fast close to ∞, there exists a unique bounded solution [48, Ch. 6.1] given that the initial condition is in $\mathcal{C}_b^2$

[4]One can notice that p_t is a Schwartz function, except [and only except] $t = 0$ where p_t is not defined. We can overcome this difficulty by defining $p_t(x) = 0$ for $t \leq 0$ and smoothing it by convolution with some $\mathcal{C}^\infty$ function with compact support. The final identity is obtained by letting the smoothing function approach point mass.

1.4.1 Stochastic integration

In this section, we will construct a two-dimensional stochastic integral that appeared in (1.9). We will repeat many arguments and constructions that can be found in the classical literature. See for example [37, 20, 19, 61, 18].

White noise is a $L^2(P)$-valued set function from $\mathscr{B}_b([0,\infty)\times\mathbb{R})$ such that for every $[0,t]\times A\in\mathscr{B}_b([0,\infty)\times\mathbb{R})$, the random variable

$$\eta([0,t]\times A)$$

is centered and Gaussian. The covariance of $\eta([0,t]\times A)$ and $\eta([0,s]\times B)$ is

$$\begin{aligned} \mathrm{E}\left[\eta([0,t]\times A)\eta([0,s]\times B)\right] &= \int_0^\infty\int_{\mathbb{R}} \mathbb{1}_{[0,t]}(\theta)\mathbb{1}_{[0,s]}(\theta)\mathbb{1}_A(x)\mathbb{1}_B(x)dxd\theta && (1.10)\\ &= t\wedge s\int_{\mathbb{R}}\mathbb{1}_{A\cap B}(x)dx. && (1.11)\end{aligned}$$

The generalized Gaussian process η exits. This follows from an application of Kolmogorov's consistency theorem in Appendix B, with $m\equiv 0$ and C given by the right-hand side of (1.11). Moreover, we have that if A and B are disjoint, then $\eta([0,t]\times A)$ and $\eta([0,s]\times B)$ are independent. We can get similar results if the time intervals entering the noise are disjoint. It will become obvious that we can think of η as *"$L^2(P)$ valued measure"*. We need a few measure theoretic properties of η:

1. *White noise is additive.* For disjoint sets $A_i\in\mathscr{B}_b([0,\infty)\times\mathbb{R})$, where i is in some finite index set I, we have

   $$\eta\left(\bigcup_{i\in I}A_i\right)=\sum_{i\in I}\eta(A_i)\quad\text{a.s.}$$

   Proof. [37, Lem. 2.3] Due to the recursive nature of this statement, we just need to show this result for $I=\{1,2\}$. Compute

   $$\mathrm{E}\left[\left(\eta(A_1\cup A_2)-\eta(A_1)-\eta(A_2)\right)^2\right]$$

   and get that the variance of a random variable $\eta(A_1\cup A_2)-\eta(A_1)-\eta(A_2)$ is zero. This shows finite additivity. □

2. *White noise is continuous from above.* Suppose that we have sets $B_i\in\mathscr{B}_b([0,\infty)\times\mathbb{R}), i\in\mathbb{N}$ such that $B_i\supset B_{i+1}$, then

   $$\eta\left(\bigcap_{i\in\mathbb{N}}B_i\right)=\lim_{i\to\infty}\eta(B_i)\quad(a.s.),$$

   where the limit on the right-hand side is taken in $L^2(P)$.

Proof. [37, Lem. 2.3] Let $B := \bigcap_{i\in\mathbb{N}} B_i$, all we need to show is that

$$\mathrm{E}\left[(\eta(B) - \eta(B_i))^2\right]$$

converges to zero as $i \to \infty$. We can write

$$\mathrm{E}\left[(\eta(B) - \eta(B_i))^2\right] = \mathrm{Leb}(B) - 2\mathrm{Leb}(B) + \mathrm{Leb}(B_i),$$

where Leb stands for the two-dimensional Lebesgue measure. Lebesgue measure is continuous from above and thus we can conclude the continuity from above of η. □

We often call η a martingale measure. This is because the stochastic process

$$M_t := \eta([0,t] \times A)$$

is a martingale for fixed $A \in \mathscr{B}_b(\mathbb{R})$ with respect to the filtration $\mathscr{F}_t$ generated by the white noise eta. Filtration $\mathscr{F}_t$ is generated by random variables of form $\eta([0,s] \times A)$ where $0 \le s \le t$ and $A \in \mathscr{B}_b(\mathbb{R})$. Or more precisely

$$\mathscr{F}_t := \sigma\left(\left\{\eta([0,s] \times A)^{-1}(B) : 0 \le s \le t, A \in \mathscr{B}_b(\mathbb{R}), B \in \mathscr{B}(\mathbb{R})\right\}\right),$$

where the exponent -1 stands for preimage.

To define a stochastic integral with respect to η, we will need to introduce an *elementary function* [37, Ch. 4.2]. Let $X \in L^2(P)$ be a $\mathscr{F}_a$ measurable random variable, $A \in \mathscr{B}_b(\mathbb{R})$ and $0 \le a \le b < \infty$, then we will call

$$\Phi(t,x) = X\mathbb{1}_{(a,b]}(t)\mathbb{1}_A(x) \tag{1.12}$$

an elementary function. We can define a stochastic integral of an elementary function as

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \Phi(s,x)\eta(dx,ds) &:= X\left(\eta([0,b\wedge t] \times A) - \eta([0,a\wedge t] \times A)\right) \qquad (1.13)\\ &= X(M_{t\wedge b}(A) - M_{t\wedge a}(A)).\end{aligned}$$

We will call a finite linear combination of elementary function a *simple function*. Intuitively, for simple function $\Psi(t,x) = \sum_{i=1}^n a_i\Phi_i(t,x)$, we will define

$$\int_0^t \int_{\mathbb{R}} \Psi(s,x)\eta(ds,dx) = \sum_{i=1}^n a_i \int_0^t \int_{\mathbb{R}} \Phi_i(s,x)\eta(ds,dx).$$

In order to extend the stochastic integral to more general class of functions, white noise η needs to be a worthy martingale measure [19, p. 6], [61, pp. 298-290], [37, pp. 18-23] which it is[5].

[5] Details might be found in any of the three provided references

From now on, we can focus on finite time interval $[0,T]$. Define norm [20, p. 20] on the space of simple functions as

$$\|\Psi\|_M^2 := \mathrm{E}\left[\int_0^T \int_{\mathbb{R}} |\Psi(s,x)|^2 \, dx ds\right],$$

and let $\mathscr{P}_M$ be the completion of the space of simple functions (up to time T) with respect to the norm $\|\cdot\|_M$. The space $\mathscr{P}_M$ will be the space of function for which we can define the stochastic integral. The following identifies some of the elements of $\mathscr{P}_M$:

1. Every $f \in L^2([0,\infty) \times \mathbb{R})$ belongs to $\mathscr{P}_M$.

2. If $f(\omega,t,x)$ is a predictable process that satisfies $\|f\|_M < \infty$, then f belongs to $\mathscr{P}_M$. Predictable processes $f(\cdot,t,x)$ are measurable with respect to the sigma algebra $\mathscr{F}_{t-} := \bigvee_{s<t} \mathscr{F}_s$. Continuous adapted processes are predictable processes.

For each $f \in \mathscr{P}_M$, we can find a sequence $\{f_i\}_{i\in\mathbb{N}}$ of simple functions such that it is Cauchy and $\|f - f_i\|_M$ converges to zero as $i \to \infty$. This immediately gives us that stochastic integral of f_i is Cauchy in $L^2(P)$ since

$$\mathrm{E}\left[\left(\int_0^T \int_{\mathbb{R}} f_i(\omega,s,x)\eta(ds,dx) - \int_0^T \int_{\mathbb{R}} f_j(\omega,s,x)\eta(ds,dx)\right)^2\right] = \|f_i - f_j\|_M^2 .$$

Thus we can define

$$\int_0^T \int_{\mathbb{R}} f(\cdot,s,x)\eta(ds,dx) := \mathrm{L}^2\text{-}\lim_{i\to\infty} \left(\int_0^T \int_{\mathbb{R}} f_i(\cdot,s,x)\eta(ds,dx)\right),$$

for any $f \in \mathscr{P}_M$.

Corollary 1.4.3 (Isometry). *For any $f \in \mathscr{P}_M$, we have*

$$\mathrm{E}\left[\left(\int_0^T \int_{\mathbb{R}} f(\omega,s,x)\eta(ds,dx)\right)^2\right] = \|f\|_M^2 .$$

Moreover, for any nonrandom $f \in L^2([0,T] \times \mathbb{R})$, we have

$$\mathrm{E}\left[\left(\int_0^T \int_{\mathbb{R}} f(\omega,s,x)\eta(ds,dx)\right)^2\right] = \|f\|_M^2 = \|f\|_{L^2([0,T]\times\mathbb{R})}^2 .$$

1.4.2 Existence and uniqueness

Now we have all the tools necessary to proceed with an interpretation of the solution to (1.6). The upcoming theorem and proof is a variation on proofs found in [61, 37, 20].

Theorem 1.4.4 (Existence and Uniqueness)**.** *There exists a stochastic process $u \in \mathscr{P}_M$ satisfying* (1.9) *up to time T, subject to*

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \mathrm{E}\left[u(t,x)^2\right] < \infty. \tag{1.14}$$

Moreover, this process in unique up to indistinguishability.

Two stochastic processes might be *'similar'* in at least two ways. We often talk about modification and indistinguishability.

Definition 1.4.5 (Modification)**.** *[51, I.1.7] Two processes X and Y defined on the same probability space are said to be modifications (versions[6]) of each other if*

$$X_t = Y_t \ \textit{a.s.}$$

for each $t \in I$. Or in other words

$$\mathrm{P}\left(X_t = Y_t\right) = 1, \forall t \in I.$$

Definition 1.4.6 (Indistinguishable)**.** *[51, I.1.7] Two processes X and Y defined on the same probability space are said to indistinguishable if*

$$X_t = Y_t \ \textit{for every } t \in I,$$

with probability one. Or in other words

$$\mathrm{P}\left(\forall t \in I, X_t = Y_t\right) = 1.$$

Proof of Theorem 1.4.4, due to [24]. **(Uniqueness)**

Suppose that there are $u, v \in \mathscr{P}_M$ solutions to (1.9), both subject to (1.14). That is

$$\begin{aligned} u(t,x) &= p_t * w(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))\eta(ds,dy) \\ v(t,x) &= p_t * w(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(v(s,y))\eta(ds,dy), \end{aligned}$$

for $0 \le t \le T$. Use Corollary 1.4.3 to get

$$\begin{aligned} \mathrm{E}\left[(u(t,x)-v(t,x))^2\right] &= \mathrm{E}\left[\left(\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\left(\sigma(u(s,y)) - \sigma(v(s,y))\eta(ds,dy)\right)\right)^2\right] \\ &= \mathrm{E}\left[\int_0^t\int_{\mathbb{R}} p_{t-s}^2(x-y)\left(\sigma(u(s,y)) - \sigma(v(s,y))\right)^2 dxds\right] \\ &\le \mathrm{Lip}_\sigma^2 \int_0^t \sup_{x\in\mathbb{R}}\left(e^{-\gamma s}\,\mathrm{E}\left[(u(s,x)-v(s,x))^2\right]\right) e^{\gamma s}\int_{\mathbb{R}} p_{t-s}(x)^2 dxds, \end{aligned}$$

[6] We deviate from the definition of *version* in [51]. In the present work, we will take version to be defined by Def. 1.4.5

where Lip_σ is the Lipschitz constant for σ. We also take $\gamma > 0$ to be some constant which will be specified later. Use the form of p_s (density of Gaussian r.v. with variance s) to get an estimate

$$\int_{\mathbb{R}} p_s(x)^2 dx \leq \sup_{x\in\mathbb{R}} p_s(x) \int_{\mathbb{R}} p_s(x) dx \leq (2\pi s)^{-1/2}.$$

Use the previous line to further write

$$\mathrm{E}\left[(u(t,x) - v(t,x))^2\right] \leq C_1 \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\, \mathrm{E}\left[(u(t,x) - v(t,x))^2\right]\right) \int_0^t e^{\gamma s} \frac{1}{\sqrt{t-s}} ds$$

where

$$C_1 = \mathrm{Lip}_\sigma^2\ (2\pi)^{-1/2}. \tag{1.15}$$

Multiply the quantity on the left-hand side by $e^{-\gamma s}$ to get

$$e^{-\gamma t}\, \mathrm{E}\left[(u(t,x) - v(t,x))^2\right] \leq C_1 \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\, \mathrm{E}\left[(u(t,x) - v(t,x))^2\right]\right) \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}} ds,$$

which yields the final estimate

$$\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\, \mathrm{E}\left[(u(t,x) - v(t,x))^2\right]\right) \leq C_2 \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\, \mathrm{E}\left[(u(t,x) - v(t,x))^2\right]\right), \tag{1.16}$$

for

$$C_2 = \mathrm{Lip}_\sigma^2\ (2\pi)^{-1/2} \int_0^\infty \frac{e^{-\gamma s}}{\sqrt{s}} ds. \tag{1.17}$$

By Dominated Converge Theorem, we can choose γ large enough, such that $C_2 < 1$. This gives us

$$u(t,x) = v(t,y) \text{ a.s.}$$

We will repeat a similar type of argument and technique numerous times throughout this work.

(Existence)

We will show existence using Picard iterates. Define

$$\begin{aligned} u_n(t,x) &= p_t * w(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\left(u_{n-1}(s,y)\right)\eta(ds,dy) \\ u_0(t,x) &= w(x). \end{aligned}$$

We will show that $u_n(t,x)$ converges in $L^2(P)$ for every $t \in [0,T]$ and $x \in \mathbb{R}$. Use Corollary 1.4.3 and follow similar steps as in the proof of uniqueness to get

$$
\begin{aligned}
&\mathrm{E}\left[(u_n(t,x)-u_{n-1}(t,x))^2\right] \\
&\le \mathrm{E}\left[\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 \left(\sigma(u_{n-1}(s,y))-\sigma(u_{n-2}(s,y))\right)^2 dsdy\right] \\
&\le C_1 \sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\,\mathrm{E}\left[(u_{n-1}(t,x)-u_{n-2}(t,x))^2\right]\right) \int_0^t \frac{e^{\gamma s}}{\sqrt{t-s}} ds,
\end{aligned}
$$

where C_1 is defined in (1.15). Multiply by $e^{-\gamma t}$ and take a supremum over space and time variable to obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\,\mathrm{E}\left[(u_n(t,x)-u_{n-1}(t,x))^2\right]\right) \\
&\le C_2 \sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\,\mathrm{E}\left[(u_{n-1}(t,x)-u_{n-2}(t,x))^2\right]\right) \\
&\le C_2^{n-1} \sup_{0\le t\le T}\sup_{x\in\mathbb{R}} \left(e^{-\gamma t}\,\mathrm{E}\left[(u_1(t,x)-u_0(t,x))^2\right]\right),
\end{aligned}
$$

where C_2 is also defined by the relation (1.17). For γ large enough, the constant C_2 is smaller than one and we get that $u_n(t,x)$ converges in $L^2(P)$. Define the limit to be $u(t,x)$. We can take a subsequence of u_n such that it converges almost surely to $u(t,x)$. So far, we have

$$
u(t,x) = p_t * w(x) + \lim_{n_k\to\infty} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{n_k}(s,y))\eta(s,y) \quad \text{a.s.}
$$

Interchanging the stochastic integral with the limit on the right-hand side of the previous line would conclude the proof. One simply needs to show that

$$
\mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{n_k}(s,y))\eta(ds,dy) - \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))\eta(ds,dy)\right)^2\right]
$$

converges to zero as $k \uparrow \infty$. We omit the proof as it uses the same technique described above. $\square$

CHAPTER 2

WEAK CONVERGENCE

Throughout this section, we will consider the following one-dimensional heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u_{\alpha,t}(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_{\alpha,t}(x) + \sigma(u_{\alpha,t}(x))\eta_\alpha(t,x)\ ,\ x \in \mathbb{R}\ ,\ t \geq 0, \\ u_{\alpha,0} &= w(x), \end{aligned} \tag{2.1}$$

with $\kappa > 0$ and Gaussian space time colored noise η_α [19]. The initial condition $w(x)$ is taken to be bounded and ϱ-Hölder continuous. We will also assume σ to be Lipschitz continuous; there exists $K \geq 0$ such that $|\sigma(x) - \sigma(y)| \leq K\,|x - y|$ and $|\sigma(x)| \leq K(1 + |x|)$. The noise η_α is assumed to have a particular covariance structure

$$\mathrm{E}\left[\eta_\alpha(t,x)\eta_\alpha(s,y)\right] = \delta(t-s)f_\alpha(x-y). \tag{2.2}$$

Noise η_α takes as argument sets of form $[0,t] \times A$, where $A \in \mathscr{B}(\mathbb{R})$ and has finite Lebesgue measure. Equation (2.2) is interpreted in the following sense:

$$\begin{aligned} \mathrm{E}\left[\eta_\alpha([0,t]\times A)\eta_\alpha([0,s]\times B)\right] &= \int_{\mathbb{R}^4} \mathbb{1}_{[0,t]}(\tau)\mathbb{1}_A(x)\delta(\tau-\nu)f_\alpha(x-y)\mathbb{1}_{[0,s]}(\nu)\mathbb{1}_B(y)d\tau d\nu dx dy \\ &= \int_{\mathbb{R}^3} \mathbb{1}_{[0,t]}(\tau)\mathbb{1}_{[0,s]}(\tau)\mathbb{1}_A(x)f_\alpha(x-y)\mathbb{1}_B(y)d\tau dx dy. \end{aligned} \tag{2.3}$$

One can notice that f_α cannot be any function. For example, the choice of $f_\alpha(x) = -1$ would easily give us

$$\mathrm{E}\left[\eta_\alpha([0,t]\times A)^2\right] < 0,$$

for any set A with positive Lebesgue measure. We will constrain ourselves to a class of positive definite functions.

2.1 Choice of f_α and positive definite functions

Naturally, we would like to have

$$\mathrm{E}\left[\eta_\alpha([0,t]\times A)^2\right] \geq 0.$$

Definition 2.1.1 (Positive definite function)**.** *We say that a generalized function $f \in \mathscr{S}^*$ is positive definite if for any Schwartz function $\varphi \in \mathscr{S}$, we have*

$$\int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \varphi(t)\varphi(t-x)dxdt = \int_{\mathbb{R}} f(t)(\varphi * \breve{\varphi})(t)dt \geq 0,$$

where $\breve{\varphi}(x) = \varphi(-x)$.

The previous definition is equivalent to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) f(x-y)\varphi(y)dxdy \geq 0,$$

if the generalized function f is a bounded continuous function. According to the Definition 2.1.1, Dirac's delta function is positive definite. This *'function'* is just unit point mass at zero. That is, if $f(x) = \delta_0(x)$, then

$$\int_{\mathbb{R}} \delta_0(x) \int_{\mathbb{R}} \varphi(t)\varphi(t-s)dtdx = \int_{\mathbb{R}} \varphi(t)\varphi(t-0)dt = \int_{\mathbb{R}} \varphi(t)^2 dt \geq 0.$$

Positive definite functions are deeply connected with the Fourier transform of generalized functions. The Fourier transform of generalized function f is a generalized function g which satisfies the following equation for any $\varphi \in \mathscr{S}$ [28, p. 169]

$$\int_{\mathbb{R}} f(x)\varphi(x)dx = (2\pi)^{-1} \int_{\mathbb{R}} g(\xi)\hat{\varphi}(\xi)d\xi. \tag{2.4}$$

We define $\hat{f} = g$ through (2.4). At this point it is not clear that $\hat{f}$ in (2.4) is truly a generalized function. It is obviously linear, which means that all we need to show is continuity, which is shown in [28, p. 169]. One can notice that we defined Fourier transform using Parseval's identity. For $F \in L^1(\mathbb{R})$, we will take $\hat{F}(\xi) = \int_{\mathbb{R}} e^{-2\pi\xi x} F(x)dx$.

Definition 2.1.2 (Tempered measure)**.** *[29, p. 140] We call a positive measure μ tempered if the integral*

$$\int_{\mathbb{R}} (1 + |x|^2)^{-p} d\mu(x)$$

converges for some $p \geq 0$.

Theorem 2.1.3 (Bochner-Schwartz)**.** *[29, Thm. 3,p. 157] Every positive definite generalized function $f \in \mathscr{S}^*$ is the Fourier transform of a positive tempered measure μ, that is, can be written as*

$$\int_{\mathbb{R}} f(x)\varphi(x)dx = \int_{\mathbb{R}} \hat{\varphi}(\xi)d\mu(\xi).$$

Conversely, the Fourier transform of any positive tempered measure defines a positive-definite generalized function $f \in \mathscr{S}^*$.

Thanks to the Bochner-Schwartz theorem, we can write for positive definite $f \in \mathscr{S}^*$ and $\varphi \in \mathscr{S}$

$$\begin{aligned}(f, \varphi * \breve{\varphi}) &= \int_{\mathbb{R}} f(x)(\varphi * \breve{\varphi})(x)dx = \int_{\mathbb{R}} \widehat{(\varphi * \breve{\varphi})}(\xi)d\mu(\xi) = \int_{\mathbb{R}} \hat{\varphi}(\xi)\hat{\breve{\varphi}}(\xi)d\mu(\xi) \\ &= \int_{\mathbb{R}} \hat{\varphi}(\xi)\overline{\hat{\varphi}}(\xi)d\mu(\xi) = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \, d\mu(\xi). \end{aligned} \tag{2.5}$$

If f is a true generalized function, we interpret the above integral in the generalized sense. Equation (2.5) implies that a generalized function f is positive definite iff it is the Fourier transform of a positive tempered measure. We are primarily interested in the case that the generalized function f_α in (2.2) is proportional to a Riesz Kernel; that is,

$$f_\alpha(x) \propto |x|^{-\alpha} \quad \text{for some} \quad \alpha \in (0,1). \tag{2.6}$$

First of all, we need to show that f_α in (2.6) is positive definite.

Lemma 2.1.4. *[28, p. 173] Choose and fix $\alpha \in (0,1)$. Then, the Fourier transform of*

$$g(x) = \frac{1}{|x|^\alpha}$$

is

$$\hat{g}(\xi) = c_\alpha \frac{1}{|x|^{1-\alpha}} \ , \ c_\alpha = 2\frac{\sin(\alpha\pi/2)\,\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}}.$$

From now on, we choose f_α as follows: $\forall \alpha \in (0,1)$,

$$f_\alpha(x) = c_{1-\alpha} g_\alpha(x) = \hat{g}_{1-\alpha}(x) \quad \text{so that} \quad g_\alpha(x) = \frac{1}{|x|^\alpha} \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\}. \tag{2.7}$$

It is obvious that f_α is positive definite, since it is a Fourier transform of the positive tempered measure $|x|^{-\alpha}\, dx$.

The function f_α can be thought of as an *'approximation'* to the delta function in the following special sense: We know that the one-dimensional Fourier transform of $g_{1-\alpha}$, denoted by $\hat{g}_{1-\alpha}$, is equal to f_α. We also know that the Fourier transform of a constant is δ distribution. Observe that $g_{1-\alpha}$ converges pointwise to 1 as $\alpha \uparrow 1$. We will study the solution of (2.1) as a function of α. This arises noticeably in [3, Sec. 7] where the authors have shown that $L^2(P)$ norm of $u_{\alpha,t}(x)$ converges to $L^2(P)$ norm of the solution to (2.8) as $\alpha \uparrow 1$ for every $t > 0, x \in \mathbb{R}$ and $\sigma(x) = x$.

The main question, which has motivated this work, is whether the solution of (2.1) converges [*in the appropriate sense*] to the solution of the same equation, but with white noise η instead of colored noise η_α as $\alpha \uparrow 1$. By that we mean, the solution to

$$\frac{\partial}{\partial t}u_t(x) = \frac{\kappa}{2}\Delta u_t(x) + \sigma(u_t(x))\eta(t,x)\ ,\ x \in \mathbb{R}\ ,\ t \geq 0, \tag{2.8}$$

$$u_0(x) = w(x), \tag{2.9}$$

where η denotes white noise. That is, a centered generalized Gaussian Process with covariance

$$\mathrm{E}\left[\eta(t,x)\eta(s,y)\right] = \delta(t-s)\delta(x-y).$$

We hinted by the name of this chapter that the *'appropriate sense'* is going to be weak convergence, which we will describe in the next section.

2.2 Weak convergence

We can start with a description of weak convergence for ordinary random variables. Suppose that X_n is a real valued random variable, that is $X_n : \Omega \to \mathbb{R}$. We say that X_n converges weakly to X, or $X_n \Rightarrow X$ if

$$\lim_{n\to\infty} E\left[h(X_n)\right] = \mathrm{E}\left[h(X)\right],$$

for every $h \in \mathcal{C}_b(\mathbb{R})$; see [10, p. 7]. This coincides with convergence in law; that is,

$$\mathrm{P}\left(X_n^{-1}((-\infty,x])\right) =: F_n(x) \to F(x) := \mathrm{P}\left(X^{-1}((-\infty,x]\right),$$

at each x where F is continuous. The notion of weak convergence is more interesting when the random variables in question are taking values in a space such as $\mathcal{C}([0,1])$. A typical example is Brownian motion; that is stochastic process $X : \Omega \to \mathcal{C}([0,1])$ such that (see, for example [51, p. 19] or [34, Def. 1.1, p. 47])

1. $\mathrm{P}\left(X(\omega)(0) = 0\right) = 1$,
2. $X(\cdot)(t)$ is Normally distributed with mean zero and variance t,
3. $X(\cdot)(t)$ has independent increments. In other words, for $t > s$, random variable $X(\cdot)(t) - X(\cdot)(s)$ is independent of $\mathscr{F}_s = \sigma\left(\{\omega : X(\omega)(l) \in A, l \leq s, A \in \mathscr{B}(\mathbb{R})\}\right)$.

We often write $B_t(\omega)$ instead of $X(\omega)(t)$ and B_t instead of $X(\cdot)(t)$.

Definition 2.2.1 (Weak convergence of measures)**.** *[10, p. 7] Let $\mathcal{S}$ be a metric space equipped with Borel sigma algebra. We say that probability measures* P_n *on* $\mathcal{S}$ *converge weakly to probability measure* P *if*

$$\lim_{n\to\infty}\int_{\mathcal{S}} f d\,\mathrm{P}_n = \int_{\mathcal{S}} f d\,\mathrm{P}, \tag{2.10}$$

for every bounded and continuous function f. We often write $\mathrm{P}_n \Rightarrow \mathrm{P}$ *to denote the weak convergence of* P_n *to* P*.*

Definition 2.2.2 (Weak convergence of random variables)**.** *[10, p. 25] Let X_n be random variables taking values in a metric space $\mathcal{S}$ equipped with Borel sigma algebra. We say that X_n converges weakly to X, write $X_n \Rightarrow X$ if the measures induced by X_n on $\mathcal{S}$ converge weakly to a measure induced by X.*

If X is a random variable with values in $\mathbb{R}$, then we can use Definition 2.2.1 directly. But, if $X : \Omega \to \mathcal{C}([0,1])$, then we need a little bit more machinery to verify (2.10). We will need notions of tightness and convergence of finite-dimensional distributions. Both definitions follow.

Definition 2.2.3 (Tightness)**.** *[10, p. 59] A family of probability measures* $\{\mathrm{P}_n\}_{n\in I}$ *on* $\mathcal{S}$ *is tight if for every $\epsilon > 0$, there exists a compact set K such that* $\mathrm{P}_n(K) > 1-\epsilon$ *for every $n \in I$.*

Definition 2.2.4 (Convergence of finite-dimensional distributions)**.** *[10, p. 57] Let* P *and* P_n *be probability measures on* $\mathcal{S} = \left(\mathcal{C}([0,1]^k), \mathscr{B}\left(\mathcal{C}([0,1]^k)\right)\right)$*. We say that* P_n *converges in finite dimensional distributions to* P *if*

$$\mathrm{P}_n\,\pi^{-1}_{x_1,\cdots,x_l} \Rightarrow \mathrm{P}\,\pi^{-1}_{x_1,\cdots,x_l}$$

for every $l \in \mathbb{N}$ and every $x_i \in [0,1]^k$. Every $\pi_\bullet$ is a canonical projection. That is, for every l and $x_i \in [0,1]^k$:

$$\begin{aligned}\pi_{x_1,\cdots,x_l} :\ & \mathcal{S} \to \mathbb{R}^k,\\ & f \mapsto (f(x_1),\ldots,f(x_l)).\end{aligned}$$

We would like to emphasize that $\mathrm{P}_n\,\pi^{-1}_{x_1,\cdots,x_l}$ is a probability measure on $\mathbb{R}^l$. If the underlying metric space is a space of continuous functions on a compact set, then verifying (2.10) might be quite complicated. The next theorem gives us the necessary condition for the weak convergence of measures on $\mathcal{C}([0,1]^k)$.

Theorem 2.2.5. *The sequence of probability measures* $\{\mathrm{P}_n\}_{n\in\mathbb{N}}$ *on* $\mathcal{S} = \left(\mathcal{C}([0,1]^k), \mathscr{B}\left(\mathcal{C}([0,1]^k)\right)\right)$ *is tight if and only if these two conditions hold:*

1. *For each positive* η*, there exist constants* α *and* n_0 *such that*

$$\mathrm{P}_n\left(\{x : |x(0)| \geq \alpha\}\right) \leq \eta, \ \forall n > n_0.$$

2. *For each positive* ϵ *and* η*, there exist* $\delta \in (0,1)$ *and* $n_0 \geq 1$ *such that*

$$\mathrm{P}_n\left(\{x : w_x(\delta) \geq \epsilon\}\right) \leq \eta \qquad \forall n \geq n_0,$$

where $w_x(\delta) = \sup_{|s-t|\leq\delta} |x(s) - x(t)|$.

We feel like the upcoming theorem deserves a proof, since even though it follows from the proofs [10, Thm. 7.3, p. 82] and [63], we could not find a version that suits us.

Proof. (⇒) If $\{\mathrm{P}_n\}_{n\in\mathbb{N}}$ is tight, then there exists a compact set $K \subset \mathcal{C}([0,1]^k)$, such that $\mathrm{P}_n(K) > 1 - \eta$. Property 1 follows almost immediately. Indeed let us cover K by open sets $B_{i,j} = \{x : \sup_t |x(t) - i| < j\}$ ($i \in \mathbb{Z}$ and $j \in \mathbb{N}$). Since K is compact, there is a finite subcover I such that

$$\mathrm{P}_n\left(\left\{x : |x(0)| < \max_{(i,j)\in I} |i + j|\right\}\right) \geq \mathrm{P}_n\left(\bigcup_{(i,j)\in I} B_{i,j}\right) \geq 1 - \eta.$$

Property 2 requires just slightly more work. Observe that for every fixed $\delta > 0$, the function $w_x(\delta)$ is continuous in x. This is because

$$|w_x(\delta) - w_y(\delta)| \leq 2 \sup_{t\in[0,1]^k} |x(t) - y(t)|.$$

We also have that for fixed $x \in \mathcal{C}\left([0,1]^k\right)$, the function $w_x(\delta)$ is nondecreasing. Moreover, we know that $\lim_{\delta\downarrow 0} w_x(\delta) = 0$ for every $x \in C\left([0,1]^k\right)$. Every continuous function on a compact set is equicontinuous. Dini's theorem [55, Thm. 7.13, p. 150] tells us that $w_x(\delta)$ converges to 0, uniformly for $x \in K$ as $\delta \downarrow 0$. This in turn implies that for every $\epsilon > 0$, there exist $\delta > 0$ such that

$$w_x(\delta) < \epsilon \qquad \forall x \in K, \ \text{ and } \ \mathrm{P}_n\left(\{x : w_x(\delta) < \epsilon\}\right) \geq \mathrm{P}_n\left(K\right) > 1 - \eta.$$

(⇐) For the other direction, we will follow the steps outlined in [9, Thm. 5.2, p. 13]. We can assume that properties 1 and 2 hold for $n_0 = 1$. This is because every single measure is tight and properties 1 and 2 hold. This immediately gives us property 1 and 2, valid for a finite family P_n, $n = 1, \ldots, n_0$, and thus for all $n \in \mathbb{N}$. Given $\eta > 0$, choose $\alpha > 0$ and $\delta_k > 0$ such that if

$$B = \{x : |x(0)| \geq \alpha\} \quad \text{and} \quad B_k = \{x : w_x(\delta_k) \geq 1/k\}\,,$$

then $\mathrm{P}_n(B) \leq \eta$ and $\mathrm{P}_n(B_k) \leq \eta/2^k$. The set $A := B^c \bigcap \left(\bigcap_{k\in\mathbb{N}} B_k^c\right)$ is bounded and equicontinuous. The Arzelà-Ascoli theorem tells us that every sequence in A has a convergent subsebsequence, where the limit point is in $K := \bar{A}$. This is one of the characterization of compact sets in metric space, therefore K is compact (every sequence has convergent subsequence). Because $\mathrm{P}_n\left(B \bigcup \left(\bigcup_{k\in\mathbb{N}} B_k\right)\right) \leq 2\eta$, we get $\mathrm{P}_n(K) \geq 1 - 2\eta$. $\square$

Theorem 2.2.6 (Weak convergence)**.** *[10, pp. 58-59] Suppose that the probability measures* $\{\mathrm{P}_n\}_{n\in\mathbb{N}}$ *on* $\mathcal{S} = \mathcal{C}([0,1]^k)$ *are tight and finite dimensional distributions of* P_n *converge to finite dimensional distributions of* P*, then* $\mathrm{P}_n \Rightarrow \mathrm{P}$*.*

2.3 Statement of the main theorem

Let us return to our original question of this chapter. We will state the main theorem in terms of measures corresponding to solutions. Let $\mathcal{C} = \mathcal{C}([0,T] \times [-N,N])$ denote the space of continuous functions on $[0,T] \times [-N,N] \subset \mathbb{R}^+ \times \mathbb{R}$, endowed with the supremum norm. Denote by P_α, the measure corresponding to u_α, restricted to $D = [0,T] \times [-N,N]$. That is,

$$P_\alpha(A) := \begin{cases} \mathrm{P}\{u_\alpha \in A^\circ\} & \text{for } \alpha \in (0,1), \\ \mathrm{P}\{u \in A^\circ\} & \text{for } \alpha = 1, \end{cases}$$

for any Borel set A of space $\mathcal{C}$. By A°, we denote the lifting of the set $A \subset \mathcal{C}$ to the larger space $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$; that is,

$$A^\circ = \{f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}) : f \text{ restricted to } [0,T] \times [-N,N] \text{ is in } A\}.$$

The initial condition $u_{\alpha,0}(x)$ is taken to be bounded and measurable. We will also assume σ to be Lipschitz continuous with Lipschitz constant K. Stochastic PDEs such as (2.1) have been studied in [19, 52, 4, 44, 17] and others. Here is the main theorem:

Theorem 2.3.1 (Main theorem - weak convergence)**.** *[8] The mapping* $\alpha \mapsto P_\alpha$ *is continuous in* α*, for* $\alpha \in (0,1]$*. We precisely mean that* P_α *converges weakly to* P_1 *as* $\alpha \uparrow 1$ *and* P_α *converges weakly to* P_{α_0} *as* $\alpha \to \alpha_0$ *for every* $\alpha_0 \in (0,1)$*.*

The notion of weak convergence (in Theorem 2.3.1) is the classical one that can be found in section 2.2 or [9, 10]. Theorem 2.3.1 gives us a new way of thinking about the Stochastic

Heat Equation with white noise. Instead of studying the solution to (2.8), we can study the solution to (2.1) for $\alpha \approx 1$. Also note, that the noise with Riesz kernel spatial covariance produces noise which is less regular than white noise. In particular, noise $\eta_\alpha(t,x)$ has long correlations in the spatial variable. We like to think that this *'roughness'* better captures properties of the Stochastic Heat Equation with white noise.

Before we begin the proof of Theorem 2.3.1, let us recall that [*mild*] solutions to (2.1) and (2.8) are interpreted as solutions of the following integral equations [20]:

$$u_{\alpha,t}(y) = (u_{\alpha,0} * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\alpha,s}(x))\eta_\alpha(ds,dx), \tag{2.11}$$

$$u_t(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_s(x))\eta(ds,dx), \tag{2.12}$$

where p_t is the heat kernel

$$p_t(x) = \frac{1}{\sqrt{2\pi\kappa t}} \exp\left(-\frac{x^2}{2\kappa t}\right),$$

and $*$ denotes the convolution of two functions. That is,

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

2.4 Proof of Theorem 2.3.1

We will only show the first part of the theorem in full detail. Namely, P_α converges weakly to P_1 as $\alpha \uparrow 1$. This is the worst-case scenario. The continuity in α for $\alpha \in (0,1)$ in Theorem 2.3.1 follows almost directly from the proof in this section. We will only comment on how the proof in the current section would change to accommodate for $\alpha \in (0,1)$.

The proof of the upcoming Theorem 2.4.2 uses coupling, which allows us to put both noises η_α and η on the same probability space. This idea was introduced in [17] and lets us write our noise η_α, for every $\alpha \in (0,1)$ in terms of one white noise η with covariance

$$\mathrm{E}\left[\eta(t,x)\eta(s,y)\right] = \delta(t-s)\delta(x-y).$$

The idea of coupling, or smoothing the noise in the spatial variable, is not new. The authors in [6] smoothed the noise in the spatial variable by an infinitely-differentiable function with compact support. They have showed that the solution to a smoothed equation converges to the solution of the heat equation with white noise as our smoothing function converges to δ distribution.

By "coupling" we mean that the martingale measure η_α will be defined as

$$\eta_\alpha([0,t]\times A) = \int_0^t \int_{\mathbb{R}} (\mathbb{1}_A * h_\alpha)(x)\eta(ds,dx), \tag{2.13}$$

where

$$h_\alpha(x) = c_{\frac{1-\alpha}{2}} g_{\frac{1+\alpha}{2}}(x) = \hat{g}_{\frac{1-\alpha}{2}}(x).$$

This choice of h_α produces the correct f_α in (2.2); that is,

$$f_\alpha(x) = (h_\alpha * h_\alpha)(x). \tag{2.14}$$

This is because

$$g_{1-\alpha}(\xi) = g_{\frac{1-\alpha}{2}}(\xi)\cdot g_{\frac{1-\alpha}{2}}(\xi).$$

The typical stochastic Fubini theorem (see [17, p. 492] or [37, p. 14, p. 53]), which would allow us to write (2.13), requires that $h_\alpha \in L^2(\mathbb{R})$. One might notice that $h_\alpha \notin L^2(\mathbb{R})$, although η_α is a well-defined martingale measure. We refer the reader to [17] for more details.

Before we state the Main Theorem of this section, let us state a technical Lemma and define the following norms [37]:

$$\mathcal{N}_{\gamma,k}(u) = \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\left(e^{-\gamma t}\left\|u_t(x)\right\|_{L^k(P)}\right) \quad , \quad \gamma > 1 \quad , \quad k \geq 2.$$

The norm $\|\cdot\|_{L^k(P)}$ in the definition of $\mathcal{N}_{\gamma,k}$ stands for L^k norm on a probability space. For a random variable X, it is defined as $\|X\|_{L^k(P)} = \mathrm{E}[|X|^k]^{1/k}$. We will often write $\|\cdot\|_k$ instead of $\|\cdot\|_{L^k(P)}$.

Lemma 2.4.1. *[30, 3.478] The following equality holds for $s > 0$ and $\beta \in [0,1)$*

$$\int_{\mathbb{R}} |x|^{-\beta} e^{-s4\pi^2x^2} dx = \left(\frac{1}{s4\pi^2}\right)^{-(\beta-1)/2} \Gamma(-\beta/2 + 1/2). \tag{2.15}$$

In the rest of this section, we will prove the following main theorem.

Theorem 2.4.2. *For every $k \geq 2$, we can find $\gamma \in (0,\infty)$ such that*

$$\lim_{\alpha\uparrow 1} \mathcal{N}_{\gamma,k}(u_\alpha - u) = 0.$$

Take the constant T that appears in Theorem 2.3.1 and the definition of the norm $\mathcal{N}_{\gamma,k}$ to be fixed throughout the proof. We will start our proof with Picard iterations for both noises η_α and η. That is, let us define for all $n \geq 0$

$$\begin{aligned}
u_t^{(0)}(y) &= (u_0 * p_t)(y) \\
u_{\alpha,t}^{(0)}(y) &= (u_0 * p_t)(y) \\
u_t^{(n+1)}(y) &= (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_s^{(n)}(x))\eta(ds,dx) \\
u_{\alpha,t}^{(n+1)}(y) &= (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\alpha,s}^{(n)}(x))\eta_\alpha(ds,dx).
\end{aligned}$$

Thanks to [17, Sec. 3.2], this is equivalent to the following:

$$\begin{aligned}
u_t^{(n+1)}(y) &= (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)) * \delta)(x)\eta(ds,dx) \\
u_{\alpha,t}^{(n+1)}(y) &= (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_{\alpha,s}^{(n)}(\cdot)) * h_\alpha)(x)\eta(ds,dx).
\end{aligned}$$

First, let us estimate the $L^k(P)$-norm of the difference $u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)$, of the Picard iterates. Namely, let us consider

$$\begin{aligned}
&\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \qquad (2.16)\\
&= \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Big((p_{t-s}(\cdot - y)\sigma(u_{\alpha,s}^{(n)}) * h_\alpha)(x) - (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * \delta)(x)\Big)\, \eta(ds,dx)\right|^k\right].
\end{aligned}$$

By adding and subtracting $(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * h_\alpha)(x)$ inside the integral and using the inequality $|a-b|^k \leq 2^k|a|^k + 2^k|b|^k$, we obtain

$$\begin{aligned}
&\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\
&\leq 2^k\,\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)})) * h_\alpha)(x)\eta(ds,dx)\right|^k\right] \\
&\quad + 2^k\,\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * (h_\alpha - \delta))(x)\eta(ds,dx)\right|^k\right].
\end{aligned}$$

The next series of steps will be used multiple times throughout this work. First we will use the Burkholder-Davis-Gundy inequality and the Minkowski integral inequality. The Burkholder-Davis-Gundy (BDG) inequality (see for example [37, Thm. B.1]) states that for any continuous L^2 martingale M_t and $k \geq 2$, we have $\|M_t\|_k^2 \leq 4k\,\|\langle M\rangle_t\|_{k/2}$, where

$\langle M \rangle_t$ denotes the quadratic variation of M. Applying this inequality and evaluating the quadratic variation term [20, Thm. 5.26] on both terms gives us

$$\begin{aligned}
&\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\
&\quad \leq \text{const} \cdot \mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)\mathfrak{v}_s^{(n)}(x,z) f_\alpha(x-z) p_{t-s}(z-y)\, ds dx dz\right)^{k/2}\right] \\
&\qquad + \text{const} \cdot \mathrm{E}\Bigg[\Bigg(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)(z)\sigma(u_s^{(n)}(x))\left(f_\alpha - 2h_\alpha + \delta\right)(x-z) \\
&\qquad\qquad p_{t-s}(z-y)\sigma(u_s^{(n)}(z))\, ds dx dz\Bigg)^{k/2}\Bigg], \qquad (2.17)
\end{aligned}$$

where

$$\mathfrak{v}_s^{(n)}(x,z) = (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z).$$

The Minkowski's integral inequality states that

$$\left(\int \left(\int f d\mu\right)^k d\nu\right)^{1/k} \leq \int \left(\int f^k d\nu\right)^{1/k} d\mu$$

for any σ-finite measures μ, ν and jointly measurable $(\mu \times \nu)$ positive function f. We use this inequality on the first term of (2.17) in to order to obtain the inequality,

$$\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \leq \text{const} \cdot \mathfrak{A}_{n,\alpha} + \text{const} \cdot \mathfrak{B}_{n,\alpha},$$

where

$$\mathfrak{A}_{n,\alpha} = \left(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y) v_s^{(n)}(x,z) f_\alpha(x-z) p_{t-s}(z-y)\, dx dz ds\right)^{k/2},$$

$$\begin{aligned}
\mathfrak{B}_{n,\alpha} = \mathrm{E}\Bigg[\Bigg(&\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)(z)\sigma(u_s^{(n)}(x))\left(f_\alpha - 2h_\alpha + \delta\right)(x-z) \\
&\qquad p_{t-s}(z-y)\sigma(u_s^{(n)}(z))\, dx dz ds\Bigg)^{k/2}\Bigg],
\end{aligned}$$

and

$$v_s^{(n)}(x,z) = \mathrm{E}\left[\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)\right|^{k/2}\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z)\right|^{k/2}\right]^{2/k}.$$

Ultimately, we would like to show that u_α is close to u as $\alpha \uparrow 1$.

We use the Cauchy-Schwarz inequality and then take a supremum over the term involving expectation, to estimate $\mathfrak{A}_{n,\alpha}$ as follows:

$$\mathfrak{A}_{n,\alpha} \leq \Bigg(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left| (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x) \right|^k \right]^{2/k} \cdot \int_{\mathbb{R}^2} p_{t-s}(x-y) f_\alpha(x-z) p_{t-s}(z-y) dxdzds \Bigg)^{k/2}.$$

The following identity holds

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) f_\alpha(x-y) \varphi(y) dxdy = \int_{\mathbb{R}} f_\alpha(x) (\varphi * \tilde{\varphi})(x) dx = \int_{\mathbb{R}} g_{1-\alpha}(\xi) |\mathcal{F}\varphi(\xi)|^2 d\xi, \qquad (2.18)$$

for any φ from Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly-decreasing test functions, where $\tilde{\varphi}(x) = \varphi(-x)$. This is a consequence of elementary properties of Fourier transform [19, p. 6],[29, pp. 151-152]. We can further rewrite $\mathfrak{A}_{n,\alpha}$, using identity (2.18) and the assumption that σ is Lipschitz continuous (there exists $K \geq 0$, such that $|\sigma(x) - \sigma(y)| \leq K|x-y|$ and $|\sigma(x)| \leq K(1+|x|)$), as

$$\begin{aligned}
\mathfrak{A}_{n,\alpha} &\leq \left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left| (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x) \right|^k \right]^{2/k} \int_{\mathbb{R}} g_{1-\alpha}(\xi) \left| \hat{p}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \\
&\leq K^k \left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left| (u_{\alpha,s}^{(n)} - u_s^{(n)})(x) \right|^k \right]^{2/k} \int_{\mathbb{R}} g_{1-\alpha}(\xi) \left| \hat{p}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2}.
\end{aligned}$$

Multiply by $e^{-k\gamma t}$ and optimize to obtain the $\mathcal{N}_{\gamma,k}$-norm in the estimate as follows:

$$\begin{aligned}
e^{-k\gamma t}\mathfrak{A}_{n,\alpha} &\leq K^k \left(\int_0^t e^{-2\gamma s} \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left| (u_{\alpha,s}^{(n)} - u_s^{(n)})(x) \right|^k \right]^{2/k} e^{-2\gamma(t-s)} \int_{\mathbb{R}} g_{1-\alpha}(\xi) \left| \hat{p}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \\
&\leq K^k \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} g_{1-\alpha}(\xi) \left| \hat{p}_{t-s}(\xi) \right|^2 d\xi ds \right)^{k/2} \\
&\leq K^k \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)\kappa 4\pi^2\xi^2} d\xi ds \right)^{k/2}. \qquad (2.19)
\end{aligned}$$

Later on, we will see that we can make the integral on the right-hand side of (2.19) arbitrarily small by choosing large γ.

The estimate for $\mathfrak{B}_{n,\alpha}$ uses a similar technique as the estimate for $\mathfrak{A}_{n,\alpha}$, but some extra work is required because of the term $(f_\alpha - 2h_\alpha + \delta)$, inside the integral, is not a positive function. Thanks to [19],[25, Cor. 3.4] identity (2.18) extends to a much broader class of functions. We will use this identity to bound term $\mathfrak{B}_{n,\alpha}$. Quantity $\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ almost surely, because

$$\mathrm{E}\left[\left\|\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y)\right\|^2_{L^2(\mathbb{R})}\right] \leq 2K^2(1+\mathcal{N}_{\gamma,2}(u^{(n)})^2)\left\|p_{t-s}(\cdot)\right\|^2_{L^2(\mathbb{R})} ,$$

and $\mathcal{N}_{\gamma,2}(u^{(n)})$ is bounded uniformly (in n, γ) for every $n \in \mathbb{N}$ and all $\gamma > \gamma_1$ [37, proof of Thm. 5.5]. The constant γ_1 depends on K, κ and $\sup_x w(x)$ [37, Thm. 5.5]. A similar reasoning applies for $\|\cdot\|_{L^1(\mathbb{R})}$. We can write

$$\mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)\sigma(u_s^{(n)}(x))\left(f_\alpha - 2h_\alpha + \delta\right)(x-z)p_{t-s}(z-y)\sigma(u_s^{(n)}(z))dxdzds\right)^{k/2}\right]$$
$$= \mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right].$$

Split this integral into two parts, and use the inequality $|a-b|^{k/2} \leq 2^{k/2}|a|^{k/2} + 2^{k/2}|b|^{k/2}$ to get

$$\mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}} (g_{1-\alpha} - 2g_{(1-\alpha)/2} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right]$$
$$\leq \mathrm{const}\cdot\left(\mathfrak{C}_{n,\alpha} + \mathfrak{D}_{n,\alpha}\right) ,$$

where

$$\mathfrak{C}_{n,\alpha} = \mathrm{E}\left[\left(\int_0^t\int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right]$$
$$\mathfrak{D}_{n,\alpha} = \mathrm{E}\left[\left(\int_0^t\int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right].$$

Properties of Fourier transform and Lipschitz continuity of $\sigma(x)$ give us

$$\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 \leq \left\|p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right\|^2_{L^1(\mathbb{R})}$$
$$\leq K^2\left\|p_{t-s}(\cdot - y)(1+|u_s^{(n)}(\cdot)|)\right\|^2_{L^1(\mathbb{R})} \leq K^2(2+2\left\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\right\|^2_{L^1(\mathbb{R})}). \tag{2.20}$$

for the term inside of $\mathfrak{C}_{n,\alpha}$. Splitting the term (2.20) inside of the integral into two yields

$$\mathfrak{C}_{n,\alpha} \leq \mathrm{E}\left[\left(\int_0^t\int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)K^2(2+2\left\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\right\|^2_{L^1(\mathbb{R})})d\xi ds\right)^{k/2}\right]$$
$$\leq C_\alpha + \mathrm{const}\cdot\mathrm{E}\left[\left(\int_0^t\int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\left\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\right\|^2_{L^1(\mathbb{R})} d\xi ds\right)^{k/2}\right],$$

where C_α denotes

$$C_\alpha = \text{const}\left(\int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2}.$$

Term C_α converges to zero as $\alpha \uparrow 1$, due to the Dominated Convergence Theorem. We use Minkowski's integral inequality to get

$$\begin{aligned}
\mathfrak{C}_{n,\alpha} &\le C_\alpha + \text{const}\left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|u_s^{(n)}(x)\right|^k\right]^{2/k} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2} \\
&\le C_\alpha + \text{const}\left(\int_0^t e^{-k\gamma s} \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|u_s^{(n)}(x)\right|^k\right]^{2/k} e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2} \\
&\le C_\alpha + \text{const}\cdot \mathcal{N}_{\gamma,k}(u^{(n)})^k \left(\int_0^t e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2}.
\end{aligned}$$

From the general theory of stochastic partial differential equations [37, proof of Thm. 5.5] we know that the term $\mathcal{N}_{\gamma,k}(u^{(n)})$ is bounded uniformly in n and γ for every $n \in \mathbb{N}$ and $\gamma > \gamma_2$. Where γ_2 again depends on K, κ and $\sup_{x\in\mathbb{R}} w(x)$. The integral term bounding $\mathfrak{C}_{n,\alpha}$ can be made arbitrarily small, again from the Dominated Convergence Theorem. Overall, we get that $\mathfrak{C}_{n,\alpha}$ converges uniformly in n to zero as $\alpha \uparrow 1$.

All we have left to do is find the estimate for $\mathfrak{D}_{n,\alpha}$. Add and subtract the term $\sigma(u_s(x))$ inside the Fourier transform, split into two integrals and obtain

$$\mathfrak{D}_{n,\alpha} \le \text{const}\cdot\mathfrak{D}_{n,\alpha}^{(1)} + \text{const}\cdot\mathfrak{D}_{n,\alpha}^{(2)},$$

where

$$\begin{aligned}
\mathfrak{D}_{n,\alpha}^{(1)} = \mathrm{E}\Bigg[\Bigg(\int_0^t \int_{\mathbb{R}\setminus[-1,1]} &(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\cdot \\
&\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\left(\sigma(u_s^{(n)}(\cdot)) - \sigma(u_s(\cdot))\right)\right)(\xi)\right|^2 d\xi ds\Bigg)^{k/2}\Bigg],
\end{aligned}$$

$$\mathfrak{D}_{n,\alpha}^{(2)} = \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right].$$

Term $\mathfrak{D}_{n,\alpha}^{(1)}$ converges to zero as $n \uparrow \infty$, uniformly in $\alpha \in (0,1)$. Use Plancherel's theorem and the fact that $(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)$ is bounded by a constant on $\mathbb{R}\setminus[-1,1]$, uniformly for all $\alpha \in (0,1)$ to write

$$\mathfrak{D}_{n,\alpha}^{(1)} \le \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}} \left(p_{t-s}(x-y)\left(\sigma(u_s^{(n)}(x)) - \sigma(u_s(x))\right)\right)^2 dx dt\right)^{k/2}\right].$$

From the convergence of Pickard's iterations and theory of SPDEs [37, proof of Thm. 5.5], we get convergence of $\mathfrak{D}_{n,\alpha}^{(1)}$ to zero as $n \to \infty$, uniformly in $\alpha \in (0,1)$. For $\epsilon > 0$, we can find $n_0(\epsilon)$ such that for every $n > n_0$, we have $\mathfrak{D}^{(1)}_{n,\alpha} < \epsilon/2$. If $n \le n_0$, we can find α_n such that $\mathfrak{D}^{(1)}_{n,\alpha} < \epsilon/2$ for $\alpha \in (\alpha_n, 1)$, by the Dominated Convergence Theorem. It might help to recall that $(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)$ converges pointwise to zero. If we apply the Dominated Convergence Theorem to $\mathfrak{D}^{(2)}_{n,\alpha}$, then we get that $\mathfrak{D}^{(2)}_{n,\alpha} < \epsilon/2$ for every $\alpha \in (\alpha_0, 1)$. Altogether we see that for every $\epsilon > 0$, there exists $\alpha_\epsilon := \max_{0\le n\le n_0} \alpha_n$ such that $\mathfrak{D}_{n,\alpha} < \epsilon$ for every $\alpha \in (\alpha_\epsilon, 1)$ and every $n \in \mathbb{N}$. Therefore $\mathfrak{D}_{n,\alpha}$ converges, uniformly in n, to zero as $\alpha \uparrow 1$.

We have shown that

$$\mathrm{E}\left[\left|u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)\right|^k\right] \le \mathrm{const}\cdot\mathfrak{A}_{n,\alpha} + \mathrm{const}\cdot\mathfrak{C}_{n,\alpha} + \mathrm{const}\cdot\mathfrak{D}_{n,\alpha},$$

where $\mathfrak{C}_{n,\alpha} + \mathfrak{D}_{n,\alpha}$ converges to zero as $\alpha \uparrow 1$, uniformly in n. For every $\epsilon > 0$, we can choose α_ϵ such that

$$\mathrm{E}\left[\left|u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)\right|^k\right] \le \mathfrak{A}_{n,\alpha} + \epsilon,$$

for every $\alpha \in (\alpha_\epsilon, 1)$. Multiply the previous line by $e^{-k\gamma t}$ and use (2.19) to arrive at the inequality,

$$\begin{aligned} &e^{-k\gamma t}\,\mathrm{E}\left[\left|u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)\right|^k\right] \\ &\quad\le \mathrm{const}\cdot\mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)}\int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)\kappa 4\pi^2\xi^2} d\xi ds\right)^{k/2} + \epsilon\ , \end{aligned}$$

where $\gamma \ge 1$. We use Lemma 2.4.1 to evaluate the term inside the integral. A straightforward calculation yields, for all $1 > \alpha > \alpha_\epsilon > 0$,

$$\begin{aligned} &e^{-k\gamma t}\,\mathrm{E}\left[\left|u^{(n+1)}_{\alpha,t}(y) - u^{(n+1)}_t(y)\right|^k\right] \\ &\le \mathrm{const}\cdot\mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\alpha/2} ds\right)^{k/2} + \epsilon \\ &\le \mathrm{const}\cdot\mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left(\left(\frac{1}{\gamma}\right)^{1-\alpha/2}\right)^{k/2} + \epsilon \le \mathrm{const}\cdot\mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k \left(\frac{1}{\gamma}\right)^{k/4} + \epsilon\ . \end{aligned}$$

We can take supremum over $y \in \mathbb{R}$ and $t \in [0, T]$ to get

$$\mathcal{N}_{\gamma,k}(u^{(n+1)}_\alpha - u^{(n+1)})^k \le \mathfrak{a}\cdot\mathcal{N}_{\gamma,k}(u^{(n)}_\alpha - u^{(n)})^k + \epsilon\ , \tag{2.21}$$

where $\mathfrak{a} = \mathrm{const}\,(1/\gamma)^{k/4}$. The constant in $\mathfrak{a}$ depends only on K, k and our choice of α_ϵ. It can be made explicit by tracking the constants in front of $\mathfrak{A}_{n,\alpha}$ together with a constant dependent on α_ϵ, which comes from Lemma 2.4.1. The dependency on α_ϵ comes only from the constant that appears in Lemma 2.4.1 and can be bounded from above as long as α_ϵ is bounded away from zero. We can always pick $\alpha_\epsilon > 1/2$ and get rid of the dependency in α_ϵ.

Equation (2.21) defines a convergent geometric series assuming that $\mathfrak{a} < 1$ and $\gamma > \max(\gamma_1, \gamma_2)$. With those choices in place, we have

$$\mathcal{N}_{\gamma,k}(u_\alpha^{(n+1)} - u^{(n+1)})^k \leq \mathfrak{a}^n \mathcal{N}_{\gamma,k}(u_\alpha^{(1)} - u^{(1)})^k + \sum_{i=1}^{n-1} \mathfrak{a}^i \epsilon,$$

and

$$\mathcal{N}_{\gamma,k}(u_\alpha^{(n+1)} - u^{(n+1)})^k \leq \frac{\epsilon}{1-\mathfrak{a}}.$$

Let n go to infinity to conclude the proof.

2.4.1 Continuity in $\mathcal{N}_{\gamma,k}$ norm

Our proof of Theorem 2.4.2 also implies continuity in $\mathcal{N}_{\gamma,k}$-norm for $\alpha \in (0,1)$. We will only comment on how the proof would change in section 2.4.

Theorem 2.4.3. *For every $k \geq 2$ and $\alpha_0 \in (0,1)$, we can find γ such that*

$$\lim_{\alpha \to \alpha_0} \mathcal{N}_{\gamma,k}(u_\alpha - u_{\alpha_0}) = 0.$$

The proof of Theorem 3 follows the same general direction of the proof of Theorem 2.4.2 with the following changes. We need to replace $u_t(x)$ with $u_{\alpha_0,t}(x)$ and change $(f_\alpha - 2h_\alpha + \delta)$ in the estimate for $\mathfrak{B}_{n,\alpha}$ to $\left(f_\alpha - 2f_{\frac{\alpha+\alpha_0}{2}} + f_{\alpha_0}\right)$ and change $\left(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1\right)$ to

$$\left(g_{1-\alpha} - 2g_{1-\frac{\alpha+\alpha_0}{2}} + g_{1-\alpha_0}\right).$$

We will also need an existence of γ_0 such that for every $\gamma > \gamma_0$, the norms $\mathcal{N}_{\gamma,k}(u_{\alpha_0})$, $\mathcal{N}_{\gamma,k}(u_{\alpha_0}^{(n)})$, $\mathcal{N}_{\gamma,2}(u_{\alpha_0})$ and $\mathcal{N}_{\gamma,2}(u_{\alpha_0}^{(n)})$ are finite, uniformly in $n \in \mathbb{N}$ and γ. This fact can be deduced from [17, Prep. 9.1].

One can notice, in both Theorems 2.4.2 and 2.4.3, that the coefficient $e^{-\gamma t}$ of the $\mathcal{N}_{\gamma,k}$-norm served only as a helping hand to make part of the term $\mathfrak{A}_{n,\alpha}$ in (2.19) small. Let us summarize our efforts thus far as

Corollary 2.4.4. *Define $u_{1,t}(x) \equiv u_t(x)$, then for all $\alpha_0 \in (0,1]$,*

$$\lim_{\alpha \to \alpha_0} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \mathrm{E}\left[|u_{\alpha,t}(x) - u_{\alpha_0,t}(x)|^k\right] = 0.$$

2.4.2 Convergence of finite-dimensional distributions

Among other things, Theorem 2.4.2 states that the solution u_α converges to u in $L^2(P)$ for every $t \in [0,T]$ and $x \in \mathbb{R}$. This implies the weak convergence of the finite-dimensional distributions of u_α to those of u.

The easiest way to see that is to show convergence in probability for a finite number of pairs $(t_i, x_i) \in [0,T] \times \mathbb{R}$. That would imply the weak convergence of the finite-dimensional distributions.

By Chebyshev's inequality, for every $x_i \in \mathbb{R}$ and $t_i \in [0,T]$,

$$\mathrm{P}\left(\sum_{i=1}^{l}(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2 > \epsilon^2\right) \leq \frac{\sum_{i=1}^{l} \mathrm{E}\left[(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2\right]}{\epsilon^2}. \tag{2.22}$$

This implies convergence in probability. From (2.22), we can conclude the weak convergence of the finite-dimensional distributions [10, p. 27]. Another way to see that is to use property [10, (ii), p. 26] which states that we have weak convergence of a finite-dimensional distribution if for all bounded, uniformly continuous functions $f : \mathbb{R}^l \to \mathbb{R}$,

$$\lim_{\alpha \uparrow 1} \mathrm{E}\left[f(u_{\alpha,t_1}(x_1), \ldots, u_{\alpha,t_l}(x_l))\right] = \mathrm{E}\left[f(u_{t_1}(x_1), \ldots, u_{t_l}(x_l))\right]. \tag{2.23}$$

Equation (2.23) holds since we have $L^2(\mathrm{P})$ convergence and f is continuous and bounded.

2.4.3 Estimates for Kolmogorov's continuity theorem and tightness

We will prove tightness (and thus weak convergence) from Kolmogorov's continuity theorem [37, p. 107]. Before we begin the proof of tightness, we will need the following three Lemmas.

Lemma 2.4.5. *[17, Lemma 6.4] For all $t > 0$ and $x \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} |p_t(y - x) - p_t(y)|\, dy \leq \mathrm{const} \cdot \left(\frac{|x|}{\sqrt{\kappa t}} \wedge 1\right),$$

where the implied constant does not depend on (t, x).

The next Lemma give us an estimate on the difference of densities $p_t(x)$ in the spatial variable.

Lemma 2.4.6. *For all $t, \epsilon > 0$,*

$$\int_{-\infty}^{\infty} |p_{t+\epsilon}(y) - p_t(y)|\, dy \leq \mathrm{const} \cdot \left((\log(t+\epsilon) - \log(t)) \wedge 1\right).$$

Proof. Direct computation gives us

$$\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)|\, dy = \int_{\mathbb{R}} \left| \int_t^{t+\epsilon} \dot{p}_s(y) ds \right| dy = \int_{\mathbb{R}} \left| \int_t^{t+\epsilon} \left(-\frac{1}{2s} + \frac{y^2}{2s^2\kappa} \right) p_s(y) ds \right| dy$$
$$\leq \int_t^{t+\epsilon} \int_{\mathbb{R}} \left(\frac{1}{2s} + \frac{y^2}{2s^2\kappa} \right) p_s(y) dy ds = \int_t^{t+\epsilon} \frac{1}{s} ds = (\log(t+\epsilon) - \log(t)) .$$

In addition, we have that $\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)|\, dy \leq 2$. □

Lastly, we will need an estimate that takes into account the initial condition $w(x)$.

Lemma 2.4.7. *[57, p. 314] Let w be a bounded ϱ-Hölder continuous function with $1 \geq \varrho > 0$. Then, there exists $C > 0$ such that for every $t > 0, \delta > 0, x \in \mathbb{R}, z \in \mathbb{R}$,*

$$\left| \int_{\mathbb{R}} (p_t(x-y) - p_t(z-y))\, w(y) dy \right| \leq C \cdot |x - z|^{\varrho} ,$$
$$\left| \int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_t(x-y))\, w(y) dy \right| \leq C \cdot \delta^{\varrho}.$$

2.4.3.1 Difference in the spatial variable

Let

$$I_{\alpha,t}(x) = \int_0^t \int_{\mathbb{R}} (p_{t-s}(x-z) - p_{t-s}(y-z))\, \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds, dz). \tag{2.24}$$

This is the stochastic integral for the mild solution (2.11). We will estimate the spatial and the temporal increments of the stochastic integral I in this and the next section. The estimates of the differences of the solution u_α will be obtained by combining Lemma 2.4.7 and the mentioned estimates on I.

We estimate the difference in the spatial variable by considering

$$\mathrm{E}\left[|I_{\alpha,t}(x) - I_{\alpha,t}(y)|^k \right] = \mathrm{E}\left[\left| \int_0^t \int_{\mathbb{R}} (p_{t-s}(x-z) - p_{t-s}(y-z))\, \sigma(u_{\alpha,s}(z)) \eta_\alpha(ds, dz) \right|^k \right].$$

Let

$$B_s(z) = (p_{t-s}(x-z) - p_{t-s}(y-z)) \ ,$$
$$A_s(x,y) = \sigma(u_{\alpha,s}(x)) \sigma(u_{\alpha,s}(y)).$$

We will proceed just as we did in section 2.4. We use the Burkholder-Davis-Gundy inequality, Minkowski's integral inequality, and the Cauchy-Schwarz inequality, and take the absolute value inside the integral to get

$$
\begin{aligned}
&\mathrm{E}\left[\left|I_{\alpha,t}(x)-I_{\alpha,t}(y)\right|^k\right]\\
&\leq \text{const}\cdot \mathrm{E}\left[\left|\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(z-w)B_s(z)B_s(w)A_s(x,y)dsdzdw\right|^{k/2}\right]\\
&\leq \text{const}\left|\int_0^t \sup_{x\in\mathbb{R}}\|\sigma(u_{\alpha,s}(x))\|_k^2\int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(z-w)\left|B_s(z)\right|\left|B_s(w)\right|dsdzdw\right|^{k/2}\\
&\leq \text{const}\left|\int_0^t \sup_{x\in\mathbb{R}}\|\sigma(u_{\alpha,s}(x))\|_k^2\,(f_\alpha * p_{t-s})(0)\int_{\mathbb{R}}\left|p_{t-s}(x-z)-p_{t-s}(y-z)\right|dsdz\right|^{k/2}\\
&\leq \text{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left|\int_0^t (f_\alpha * p_{t-s})(0)\int_{\mathbb{R}}\left|p_{t-s}(x-z)-p_{t-s}(y-z)\right|dzds\right|^{k/2}\\
&\leq \text{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left|\int_0^t (f_\alpha * p_{t-s})(0)\left(\frac{|x-y|}{\sqrt{\kappa(t-s)}}\wedge 1\right)\right|^{k/2},
\end{aligned}
$$

where the last inequality is due to Lemma 2.4.5. We also used the fact that

$$
\int_{\mathbb{R}} f_\alpha(z-w)\left|p_{t-s}(x-w)-p_{t-s}(y-w)\right|dw \leq 2\left(f_\alpha * p_{t-s}\right)(0). \tag{2.25}
$$

The previous line (2.25) can be easily checked by the use of the triangle inequality and maximization over variables. The inequality $r\wedge 1\leq r^{2a}$ for $a\in(0,1/2)$ gives us

$$
\begin{aligned}
&\mathrm{E}\left[\left|I_{\alpha,t}(x)-I_{\alpha,t}(y)\right|^k\right]\\
&\qquad\leq \text{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left|x-y\right|^{ak}\left|\int_0^t (f_\alpha * p_{t-s})(0)\cdot(t-s)^{-a}ds\right|^{k/2}.
\end{aligned} \tag{2.26}
$$

It remains to show that the integral on the right-hand side is bounded for all $\alpha\in(\alpha_0,1)$, $\alpha_0>0$. To show this, we will need the explicit form of f_α. The requisite result is stated in the next Lemma.

Lemma 2.4.8. *For every $1>\alpha>\alpha_0>0$,*

$$
f_\alpha * p_s(0)\leq \text{const}\cdot s^{-\alpha/2},
$$

where the constant depends only on our choice of α_0.

Proof. By direct computation and (2.15), we get

$$
(f_\alpha * p_s)(0) = c_{1-\alpha}\int_{\mathbb{R}}\frac{1}{|x|^\alpha}p_s(x)dx = 2\frac{\sin\left(\frac{(1-\alpha)\pi}{2}\right)\Gamma(\alpha)}{(2\pi)^\alpha}2^{-\alpha/2}\Gamma\left(\frac{1-\alpha}{2}\right)s^{-\alpha/2}\pi^{-1/2}.
$$

The boundedness of the constant

$$
2\frac{\sin\left(\frac{(1-\alpha)\pi}{2}\right)\Gamma(\alpha)}{(2\pi)^\alpha}2^{-\alpha/2}\Gamma\left(\frac{1-\alpha}{2}\right)\pi^{-1/2}
$$

can be concluded from Euler's reflection formula ($\Gamma(1-z)\Gamma(z)=\pi/\sin(\pi z)$) for $z=(1-\alpha)/2$. $\square$

Because of Lemma 2.4.8, the integral on the right-hand side of (2.26) is finite as long as $\alpha/2 + a < 1$. Since $\alpha \in (0,1)$, we can always take $a \in (0, 1/2)$.

In [17, proof of Prep. 6.5], the authors also get (2.26), but some extra effort is required to show that (2.26) holds with one constant on the right-hand side for $\alpha \in (\alpha_0, 1), \alpha_0 > 0$. That is, we can find a constant independent of α.

2.4.3.2 Difference in the time variable

The difference in the time variable is going to be, for $\delta > 0$,

$$
\begin{aligned}
&\mathrm{E}\left[|I_{\alpha,t+\delta}(x) - I_{\alpha,t}(x)|^k\right] \\
&\quad = \text{const} \cdot \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} \left(p_{t+\delta-s}(x-z) - p_{t-s}(x-z)\right)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right] \\
&\qquad\qquad + \text{const} \cdot \mathrm{E}\left[\left|\int_t^{t+\delta} \int_{\mathbb{R}} p_{t+\delta-s}(x-z)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right].
\end{aligned}
$$

In order to estimate the second integral, we can use the same technique as in the case of the spatial variable and write

$$
\begin{aligned}
&\mathrm{E}\left[\left|\int_t^{t+\delta} \int_{\mathbb{R}} p_{t+\delta-s}(x-z)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right] \\
&\leq \text{const}\left(\int_t^{t+\delta} \sup_x \mathrm{E}\left[|\sigma(u_{\alpha,s}(x))|^k\right]^{2/k} \int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} |\hat{p}_{t+\delta-s}(\xi)|^2 \, d\xi ds\right)^{k/2} \\
&\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_t^{t+\delta} (t+\delta-s)^{-\alpha/2} ds\right)^{k/2} \\
&\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\, |\delta|^{k(2-\alpha)/4} \leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\, |\delta|^{k/4}.
\end{aligned}
$$

The estimate for the first integral will be

$$
\begin{aligned}
&\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} \left(p_{t+\delta-s}(x-z) - p_{t-s}(x-z)\right)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right] \\
&\leq \text{const}\left(\int_0^t \sup_{x\in R} \mathrm{E}\left[|\sigma(u_{\alpha,s}(x))|^k\right]^{2/k} (f_\alpha * p_{t-s})(0) \int_{\mathbb{R}} |p_{t+\delta-s}(z) - p_{t-s}(z)|\, dz ds\right)^{k/2} \\
&\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_0^t (f_\alpha * p_{t-s})(0) \int_{\mathbb{R}} |p_{t+\delta-s}(z) - p_{t-s}(z)|\, dz ds\right)^{k/2} \\
&\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_0^t s^{-1/2}\left(\log(s+\delta) - \log(s)\right) ds\right)^{k/2} \\
&\leq \text{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(4\sqrt{\delta}\,\mathrm{atan}\left(\sqrt{\frac{t}{\delta}}\right) + 2\sqrt{t}\log(1+\delta/t)\right)^{k/2},
\end{aligned}
\tag{2.27}
$$

by using a similar technique as in the case for the spatial variable and Lemma 2.4.6. The first step in (2.27) uses that

$$(f_\alpha * |p_{t+\delta-s} - p_{t-s}|)(z) \le f_\alpha * p_{t+\delta-s}(0) + f_\alpha * p_{t-s}(0) \le 2 f_\alpha * p_{t-s}(0).$$

The last step in (2.27) can be verified by differentiating the function $4\sqrt{\delta}\,\mathrm{atan}(\sqrt{t/\delta}) + 2\sqrt{t}\log(1+\delta/t)$. The inequality $\log(1+\zeta) < \sqrt{\zeta}$, for all $\zeta > 0$, gives us

$$\mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} \left(p_{t+\delta-s}(x-z) - p_{t-s}(x-z)\right) \sigma(u_{\alpha,s}(z))\eta(ds,dz)\right|^k\right] \\ \le \text{const}\left(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k\right)\delta^{k/4}. \quad (2.28)$$

We can combine both estimates (2.27) and (2.28) to finally get

$$\mathrm{E}\left[|I_{\alpha,t+\delta}(x) - I_{\alpha,t}(x)|^k\right] = \text{const}\cdot\left(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k\right)\delta^{k/4}\ . \quad (2.29)$$

Estimates similar to 2.4.3.2 and 2.4.3.1 can be found in numerous places in the literature, for example [58]. The authors in [58] use a different technique and investigate noise with more general covariance structure. We do not know of continuity estimates which take into account α as a variable, thus our estimates in sections 2.4.3.2 and 2.4.3.1 are novel in that sense.

2.4.4 Kolmogorov's continuity theorem and tightness

Let us mention that $\mathcal{N}_{k,\gamma}(u_\alpha)$ is bounded uniformly in $\gamma, \gamma > 0$, and $\alpha \in (\alpha_0, 1)$ where $\alpha_0 > 0$. This follows from Corollary 2.4.4 and it is important for bounds on differences in sections 2.4.3.2 and 2.4.3.1. We have that for every $1 > \alpha > \alpha_0 > 0$ and $(s,x),(t,y)$ from $D := [0,T]\times[-N,N] \subset \mathbb{R}_0^+ \times \mathbb{R}$, the following holds for $k \ge 2$:

$$\mathrm{E}\left[|u_{\alpha,s}(x) - u_{\alpha,t}(y)|^k\right] \le \text{const}\,|x-y|^{ka} + \text{const}\,|t-s|^{kb}\,,$$

where $a \in (0, \frac{1}{2}\wedge\varrho)$ and $b \in (0, \frac{1}{4}\wedge\varrho)$, thanks to our estimates (2.29) and (2.26) from sections 2.4.3.1,2.4.3.2 and Lemma 2.4.7. Let $\rho(t,x) = |x|^a + |t|^b$. Kolmogorov's continuity theorem implies that there is a modification of $u_{\alpha,s}(x)$ such that

$$\mathrm{E}\left[\sup_{(s,x),(t,y)\in D}\left|\frac{u_{\alpha,s}(x) - u_{\alpha,t}(y)}{\rho(s-t,x-y)^q}\right|^k\right] < \Lambda < +\infty \quad (2.30)$$

for every $\alpha \in (\alpha_0, 1)$ and $q \in (0, 1 - H/k)$ where $H = 1/a + 1/b$, see for example [37, p. 107]. By Markov's inequality and (3.19), we can write

$$\mathrm{P}\left\{\sup_{\substack{(s,x),(t,y)\in D\\ \rho(s-t,x-y)<\delta}} |u_{\alpha,s}(x) - u_{\alpha,t}(y)| > \epsilon\right\} < \frac{\Lambda}{\epsilon^k}\delta^{kq}\ ,$$

which implies

$$\lim_{\delta\to 0} \sup_{\alpha\in(\alpha_0,1)} \mathrm{P}\left\{ \sup_{\substack{(s,x),(t,y)\in D\\ \rho(s-t,x-y)<\delta}} |u_{\alpha,s}(x) - u_{\alpha,t}(y)| > \epsilon \right\} = 0 \ , \tag{2.31}$$

for every $\epsilon > 0$. We established both convergence of the finite-dimensional [63, Thm. 2 (i)] distributions and tightness (2.31) for the measures $\{P_\alpha\}$ on $\mathcal{C}$ [63, Thm. 2 (ii)]. The tightness of P_α can be seen by adapting [10, Thm. 7.3] to a setting of two-dimensional continuous functions with compact domain and supremum norm. We can conclude [63, Thm. 2] that the measure P_α corresponding to u_α restricted to D converges weakly as $\alpha \uparrow 1$ to a measure P_1 corresponding to u restricted to D. We can also conclude the weak convergence of P_α to P_1 as $\alpha \uparrow 1$ by adapting [10, Chapter 2] to the two-dimensional setting, which is done in Theorem 2.2.5

CHAPTER 3

EXISTENCE

Let us consider the following stochastic differential equation on a circle,

$$\partial_t u = \mathcal{L}u + \sigma(u)d\eta, , \tag{SHE}$$
$$u_0(x) = g(x).$$

The operator $\mathcal{L}$ in (SHE) is the generator of a Lévy process on a circle of size 2π, and g is taken to be a nonnegative continuous function. The question we will answer in this chapter is how fast can σ grow in order for (SHE) to have a solution. We will assume that σ is locally Lipschitz, $\sigma(0) = 0$, and that we have the following condition on σ:

$$|\sigma(x)| \leq a\,|x|^{\gamma} + b \quad \text{for all} \quad x \in \mathbb{R}. \tag{EX}$$

We will use condition (EX) to show the existence of a solution to (SHE). The constants a, b are positive and without loss of generality, we can assume that $a = 1$ and $b = 0$ and formulate our results in terms of the exponent γ.

3.1 Introduction and statement of main theorems

Let $\Theta := [0, 2\pi]$ with identified endpoints. We can think of a Lévy process on Θ as a Lévy process on $\mathbb{R}$ which is *'wrapped'* around Θ. In other words, we map a position on the real line to a position on a circle of length 2π. Recall the Lévy-Khintchine formula [59, Thm. 8.1]. If X_t is a Lévy process on $\mathbb{R}$ that starts at zero, then

$$\mathrm{E}\left[e^{i\xi X_t}\right] = \mathrm{E}\left[e^{i\xi X_1}\right]^t = e^{-t\psi(\xi)}$$

and

$$\psi(\xi) = ia\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left(1 - e^{i\xi z} + i\xi z \mathbb{1}_{(-1,1)}(z)\right) m(dz),$$

where $m(dz)$ is the Lévy measure of X_t and ψ is the Lévy exponent. In this work, we will consider the following assumptions on ψ:

(A1) $\lim_{\xi\to\pm\infty} \psi(\xi)/|\xi|^{\beta} = +\infty$ for some fixed $\beta > 1$,

(A2) ψ is infinitely differentiable everywhere except at distinct points $\xi_1, \ldots, \xi_M$. Moreover, in the neighborhood of each ξ_j, we can write

$$\psi(\xi) = c_j \, |\xi - \xi_j|^{\alpha_j} + h_j(\xi),$$

where $\alpha_j \in (1, 2)$ and h_j is infinitely differentiable

The size of the constants c_j in (A2) is not crucial to our analysis. We will take [*without loss of generality*] $c_j \equiv 1$. Many of our examples arose from Lévy measures for symmetric α-stable processes. Table 3.1 shows some Lévy exponents that satisfy conditions (A1), (A2) and (A3).

To show existence, we will *'cut'* the function σ at some level N, and we will follow similar steps to those in [40]. The new system will be

$$\begin{aligned} \partial_t u_N(t,x) &= \mathcal{L} u_N(t,x) + \sigma(u_N \wedge N)\eta(t,x), \\ u_N(0,x) &= g(x). \end{aligned} \tag{CAPN}$$

Note that the solution to (CAPN) and (SHE) are identical until the stopping time

$$\mathfrak{t}_N := \inf\{t > 0 : \sup_x |u(t,x)| \geq N\}, \tag{3.1}$$

that is the time when u reaches level N for the first time. Moreover, the solution u is unique up to $\mathfrak{t}(u)$ and we have that

$$u(t,x) = \lim_{N \to \infty} u_N(t,x), \text{ a.s. for } 0 \leq t \leq \mathfrak{t}(u) \text{ and } x \in [0, 2\pi].$$

The uniqueness follows from the strong (path-wise) uniqueness of the solution to (CAPN) which we will state in Theorem 3.5.7. Since $u(t \wedge \mathfrak{t}_N(u), x)$ solves (CAPN), up to time $\mathfrak{t}_N(u)$ we know that the solution to (CAPN) is unique.

We would like to ultimately show that $\mathfrak{t}_N \to \infty$ almost surely if σ satisfies bound (EX) for some suitable exponent γ. The mild form of the solution to (CAPN) is given by

$$u_N(t,x) = \int_0^{2\pi} p_t(x-y)g(y)dy + \int_0^t \int_0^{2\pi} p_{t-s}(x-y)\sigma(u_N(s,y) \wedge N)\eta(s,y). \tag{3.2}$$

The main theorem of this chapter follows.

Theorem 3.1.1 (Existence)**.** *The solution to* (SHE) *on a circle exists almost surely for all time, that is* $\mathfrak{t}_N \to \infty$ *almost surely if* σ *is bounded by* (EX) *with* $1 \leq \gamma < 1 + (\beta - 1)/2$ *and* $\mathcal{L}$ *is a generator of a Lévy process that satisfies (A1)-(A3).*

Long-time existence is known when $\mathcal{L} = \Delta = \partial^2/\partial x^2$ [the generator of Brownian motion] and $\gamma < 3/2$. This result is due to Carl Mueller [40], [20, p. 131].

By a solution to (SHE), we mean the mild solution [61, 19, 20] given by

$$u(t,x) = \int_0^{2\pi} p_t(x-y)g(y)dy + \int_0^t \int_0^{2\pi} p_{t-s}(x-y)u(s,y)\eta(ds,dy),$$

where η denotes the space time white noise. In other words, a noise with covariance

$$\mathrm{E}\left[\eta(t,x)\eta(s,y)\right] = \delta(t-s)\delta(x-y),$$

with δ being Dirac's delta mass. The function $p_t(x)$ is the transition density of the Lévy process *[starting at 0]* wrapped around a circle. At this point, it is not even clear that such a density exists; we will see that this is indeed the case.

First, will establish some key density estimates for p_t. We use many techniques from [40]; most notably we will establish [40, Lemma 2.1] for (SHE).

3.2 Density estimates and large-x asymptotics

We will need a few estimates and some asympotics on the density p_t and the density $\tilde{p}_t$. Let $\tilde{p}$ be the transition density of the Lévy process on $\mathbb{R}$. The density is given by [inverse] Fourier transform

$$\tilde{p}_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x - t\psi(\xi)} d\xi. \tag{3.3}$$

We will always denote the density of *'unwrapped'* process X_t with tilde. Both conditions (A1) and (A2) on Lévy exponent guarantees that $e^{-t\psi(\xi)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and there exists transition density for process X_t on $\mathbb{R}$ [59, Prep. 2.5(xii)].

The density of the Lévy process wrapped around the circle of size 2π is

$$p_t(x;2\pi) = p_t(x) = \sum_{k\in\mathbb{Z}} \tilde{p}_t(x+2\pi k)\ ,\ x \in [0,2\pi]. \tag{3.4}$$

For a circle of size Λ, the density $p_t(x;\Lambda)$ is going to be

$$p_t(x;2\pi) = p_t(x) = \sum_{k\in\mathbb{Z}} \tilde{p}_t(x+\Lambda k)\ ,\ x \in [0,\Lambda].$$

The density p_t takes one extra parameter, separated with a semicolon, which is the size of the domain. The domain size will play a crucial role in Chapter 4, but it will be fixed in this section. We will simply write $p_t(x)$ instead of $p_t(x;\Lambda)$ if the domain size is either $\Lambda = 2\pi$ or if the domain size is clearly determined by the context.

At this point, it is not even clear whether p_t is finite for each $x \in [0, 2\pi]$. We will show that this is indeed the case, we also establish some continuity results for p_t in both x and t, $T > t > 0$, where $T \in (0, 1]$. The finiteness of $p_t(x)$ will follow from the decay of $\tilde{p}_t(x)$ as $x \to \pm\infty$. This decay property is a direct consequence of properties of the Lévy exponent ψ.

Condition (A2) implies that we can write

$$\begin{aligned}\psi(\xi) &= \sum_{j=1}^{M} \frac{|\xi - \xi_j|^{\alpha_j}}{1 + (\xi - \xi_j)^4} + h(\xi)\\ &= \sum_{j=1}^{M} \left(\frac{|\xi - \xi_j|^{\alpha_j}}{1 + (\xi - \xi_j)^4} + |\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right) + h(\xi) - \sum_{j=1}^{M} |\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right),\end{aligned}$$

where $0 < \bar{\beta} < \beta$ and $h \in C^4(\mathbb{R})$. By the elementary properties of Fourier transforms, this in turn implies that

$$\begin{aligned}\tilde{p}_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} & e^{-i\xi x} e^{-th(\xi) + t\sum_{j=1}^{M} |\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)}\\ & \cdot \prod_{j=1}^{M} \exp\left(-t \frac{|\xi - \xi_j|^{\alpha_j}}{1 + (\xi - \xi_j)^4} - t\,|\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right) d\xi\\ = (2\pi)^M & \cdot q_t^{(0)} * q_t^{(1)} * \cdots * q_t^{(M)}(x)\ ,\end{aligned}$$

where

$$q_t^{(0)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \exp\left(-th(\xi) + t \sum_{j=1}^{M} |\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right) d\xi, \tag{3.5}$$

$$q_t^{(j)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \exp\left(-t \frac{|\xi - \xi_j|^{\alpha_j}}{1 + (\xi - \xi_j)^4} - t\,|\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right) d\xi, \tag{3.6}$$

for $j \in \{1, \ldots, M\}$. Our choice of adding and subtracting the tanh term is due to nice contour integrability. Now we will present our last assumption followed by several lemmas and a statement about asymptotics of $\tilde{p}$.

(A3) We will assume that,

$$\sup_{0<t\le T} \left\| \left(\frac{d^3}{d\xi^3} \exp\left(-th(\xi) + t \sum_{j=1}^{M} |\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right)\right)\right\|_{L^1(\mathbb{R})} < +\infty.$$

Equivalently, we require the L^1-norm of the third derivative is uniformly bounded in t for $0 < t \le T$.

Lemma 3.2.1. *Let $f(x), g(x)$ be a bounded functions such that*

$$f(x) = \mathcal{O}\left(|x|^{-\mu}\right), g(x) = \mathcal{O}\left(|x|^{-\nu}\right) \text{ as } x \to \pm\infty,$$

for some $\mu, \nu > 1$. Then

$$f * g(x) = \mathcal{O}\left(|x|^{-\gamma}\right) \text{ as } x \to \pm\infty,$$

where $\gamma = \min(\mu, \nu)$.

Proof. We can write

$$|f * g(x)| \leq \int_{|y|<\frac{|x|}{2}} |f(x-y)g(y)|\, dy + \int_{|y|>\frac{|x|}{2}} |f(x-y)g(y)|\, dy \leq C_1 |x|^{-\mu} + C_2 |x|^{-\nu}.$$

□

Lemma 3.2.2. *Let $q_t^{(j)}(x)$ be defined as in (3.6). Then for every fixed T,*

$$\sup_{0<t\leq T} \left|q_t^{(j)}(x)\right| = \mathcal{O}(|x|^{-(1+\alpha_j)}) \text{ as } x \to \pm\infty.$$

Proof. We will only show the case when $x \to \infty$, since the case $x \to -\infty$ is identical. Without the loss of generality, we can consider $x > 0$. In order to estimate

$$q_t^{(j)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \exp\left(-t\frac{|\xi - \xi_j|^{\alpha_j}}{1 + (\xi - \xi_j)^4} - t\,|\xi - \xi_j|^{\bar{\beta}} \tanh\left((\xi - \xi_j)^4\right)\right) d\xi,$$

We will compute the distance of $q^{(j)}$ to the following term

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \exp(-(\xi - \xi_j)^2) d\xi = e^{-i\xi_j x} f(x),$$

where f is the density function of a centered normal distribution with variance 2. This distance is precisely,

$$\left|q_t^{(j)}(x) - e^{-i\xi_j x} f(x)\right| \leq 2 \left|\int_0^\infty e^{-i\xi x} \left(\exp\left(-t\frac{\xi^{\alpha_j}}{1+\xi^4} - t\xi^{\bar{\beta}} \tanh(\xi^4)\right) - \exp\left(-\xi^2\right)\right) d\xi\right|. \tag{3.7}$$

We can use contour integration, with the contour depicted in Figure 3.1 ($r \to 0, R \to \infty$) and angle $\varphi = -\pi/12$. The choice of φ is arbitrary as long as it is small and negative. Equation (3.7) will now become

$$\begin{aligned} \left|q_t^{(j)}(x) - e^{-i\xi_j x} f(x)\right| &\leq 2 \left|\int_0^\infty e^{-i\xi x e^{i\varphi}} \Upsilon d\xi\right| \qquad (3.8) \\ &\leq \int_0^\epsilon \left|e^{-i\xi x e^{i\varphi}} \Upsilon\right| d\xi + \int_\epsilon^\infty \left|e^{-i\xi x e^{i\varphi}} \Upsilon\right| d\xi, \end{aligned}$$

where

$$\Upsilon = \exp\left(-t\frac{\xi^{\alpha_j} e^{i\varphi\alpha_j}}{1+\xi^4 e^{4i\varphi}} - t\xi^{\bar{\beta}} e^{i\varphi\bar{\beta}} \tanh(\xi^4 e^{4i\varphi})\right) - \exp\left(-\xi^2 e^{2i\varphi}\right).$$

Use the leading-order term in the expansion of Υ to bound the first term in (3.8) by

$$\int_0^{\epsilon} \left| e^{-i\xi x e^{i\varphi}} \Upsilon \right| d\xi \leq \text{const} \cdot t \int_0^{\epsilon} e^{\xi x \sin(\varphi)} \xi^{\alpha_j} d\xi \leq \text{const} \cdot t \int_0^{\infty} e^{\xi x \sin(\varphi)} \xi^{\alpha_j} d\xi = \text{const} \frac{t}{x^{1+\alpha_j}}.$$

The purpose of adding and subtracting the normal density is to remove the constant term in the expansion of Υ. The bound on the second integral in (3.8) is

$$\int_{\epsilon}^{\infty} \left| e^{-i\xi x e^{i\varphi}} \Upsilon \right| d\xi \leq \int_{\epsilon}^{\infty} e^{\xi x \sin(\varphi)} d\xi \leq \text{const} \frac{e^{-\epsilon x}}{x}.$$

We can combine the above results to obtain

$$\left| q_t^{(j)}(x) - e^{i\xi_j x} f(x) \right| = \mathcal{O}(|x|^{-1-\alpha_j}) \quad \text{as} \quad x \to \pm\infty.$$

Since f decays exponentially as x gets large, the preceding concludes our proof.

□

Proposition 3.2.3 (Asymptotics of $\tilde{p}$). *Let $\tilde{p}$ be the density of a Lévy process on $\mathbb{R}$ satisfying (A1)-(A3) and $\alpha = \min_j(\alpha_j)$. Then,*

$$\sup_{0<t\leq T} \tilde{p}_t(x) = \mathcal{O}(|x|^{-(1+\alpha)}) \text{ as } x \to \pm\infty,$$

for $t \in (0, T]$ and large, fixed T.

Proof. Define

$$\zeta(\xi) = \frac{d^3}{d\xi^3} \exp\left(-th(\xi) + t\sum_{j=1}^{M} |\xi - \xi_j|^{\bar{\beta}} \tanh((\xi - \xi_j)^4)\right) \in L^1(\mathbb{R}).$$

Since $h \in L^1(\mathbb{R})$, the properties of the Fourier transform and (A3) together tell us that

$$q_t^{(0)}(x) \leq \text{const} \frac{1}{x^3} \left| \int_{\mathbb{R}} e^{-ix\xi} \zeta(\xi) d\xi \right| \leq \text{const} \frac{1}{x^3} \int_{\mathbb{R}} |\zeta(\xi)| \, d\xi \leq \text{const} \frac{1}{x^3}. \tag{3.9}$$

The result of the proposition can be concluded from Lemmas 3.2.1 and 3.2.2. □

3.2.1 Density estimates

We will need a few density estimates. Write down the form of $\tilde{p}$ in (3.3) and take the absolute value inside the integral to obtain

$$\sup_{x\in\mathbb{R}} \tilde{p}_t(x) \leq \text{const}\left(1+\int_{|\xi|>R} e^{-tc|\xi|^\beta} d\xi\right) \leq \text{const}\left(1+\int_0^\infty e^{-tc|\xi|^\beta} d\xi\right) \leq \text{const}\cdot t^{-1/\beta},$$

for every $t\in(0,T]$ where T is fixed. The difference in the spatial variable can be bounded by

$$\begin{aligned}
|\tilde{p}_t(x+\epsilon)-\tilde{p}_t(x)| &= \left|\frac{1}{2\pi}\int_{-\infty}^\infty e^{-ix\xi}e^{-t\psi(\xi)}\left(e^{-i\epsilon x}-1\right)d\xi\right| \leq \frac{1}{2\pi}\int_{-\infty}^\infty e^{-t\psi(\xi)}\left(1-\cos(\epsilon\xi)\right)d\xi \\
&\leq \frac{1}{2\pi}\int_{-\infty}^\infty e^{-t\psi(\xi)}\epsilon^2\xi^2 d\xi \leq \text{const}\cdot\epsilon^2\left(R^2+\int_R^\infty e^{-t\xi^\beta}\xi^2 d\xi\right) \\
&\leq \text{const}\frac{\epsilon^2}{t^{3/\beta}}, \qquad (3.10)
\end{aligned}$$

and the bound on difference in time is, for $\epsilon>0$

$$\begin{aligned}
|\tilde{p}_{t+\epsilon}(x)-\tilde{p}_t(x)| &= \frac{1}{2\pi}\left|\int_{-\infty}^\infty e^{-ix\xi-(t+\epsilon)\psi(\xi)}-e^{-ix\xi-t\psi(\xi)}d\xi\right| \leq \frac{1}{2\pi}\int_{-\infty}^\infty e^{-t\psi(\xi)}(1-e^{-\epsilon\psi(\xi)})d\xi \\
&\leq \frac{1}{2\pi}\int_{-\infty}^\infty e^{-t\psi(\xi)}\epsilon\psi(\xi)d\xi = \text{const}\cdot\epsilon\left(1+\int_R^\infty e^{-t\xi^\beta}\xi^\beta d\xi\right) \\
&\leq \text{const}\cdot\frac{\epsilon}{t^{(1+1/\beta)}}. \qquad (3.11)
\end{aligned}$$

3.3 Existence and uniqueness of the solution to (CAPN)

Existence and uniqueness for equation of type (CAPN) is known in the case when $\mathcal{L}$ is a generator of α-stable process. For example, each of references [12, 21, 24] can be applied for special cases of $\mathcal{L}$. It is worth mentioning that each of the above references deal with the case where spatial domain is $\mathbb{R}$. In this section, we will show existence and uniqueness of solution to (CAPN), given that $\mathcal{L}$ satisfies (A1)-(A3).

Theorem 3.3.1. *There exist a mild solution* (3.2) *to* (CAPN).

Proof. The proof follows the same steps as the proof of Theorem 1.4.4. Define,

$$\begin{aligned}
u_N^0(t,x) &= \int_0^{2\pi} p_t(x-y)f(y)dy, \\
u_N^k(t,x) &= \int_0^{2\pi} p_t(x-y)f(y)dy + \int_0^t\int_0^{2\pi} p_{t-s}(x-y)\sigma(u_n^{k-1}(t,y)\wedge N)\eta(ds,dy).
\end{aligned}$$

We will show existence of solution by showing convergence of Pirard iterates. The difference of $u_N^k(t,x) - u_N^{k-1}(t,x)$ is going to be

$$
\begin{aligned}
&\mathrm{E}\left[\left|u_N^k(t,x) - u_N^{k-1}(t,x)\right|^2\right] \\
&= \mathrm{E}\left[\int_0^t \int_0^{2\pi} p_{t-s}(x-y)^2 \left(\sigma(u_N^{k-1}(t,y)\wedge N) - \sigma(u_N^{k-2}(t,y)\wedge N)\right)^2 dsdy\right] \\
&\le \mathrm{Lip}_{\sigma,N}^2 \int_0^t \int_0^{2\pi} p_{t-s}(x-y)^2\, \mathrm{E}\left[\left(u_N^{k-1}(t,y) - u_N^{k-2}(t,y)\right)^2\right] dsdy \\
&\le \mathrm{Lip}_{\sigma,N} \cdot \mathcal{N}_{\gamma,2}\left(u_N^{k-1} - u_N^k\right)^2 \int_0^t e^{2\gamma s} \int_0^{2\pi} p_{t-s}(x-y)^2 dyds.
\end{aligned}
$$

Multiply previous inequality by $e^{-2\gamma t}$ and obtain

$$
\begin{aligned}
&e^{-2\gamma t}\, \mathrm{E}\left[\left|u_N^k(t,x) - u_N^{k-1}(t,x)\right|^2\right] \\
&\le \mathrm{Lip}_{\sigma,N} \cdot \mathcal{N}_{\gamma,2}\left(u_N^{k-1} - u_N^k\right)^2 \int_0^t e^{-2\gamma(t-s)} \int_0^{2\pi} p_{t-s}(x-y)^2 dyds \\
&\le \mathrm{Lip}_{\sigma,N} \cdot \mathcal{N}_{\gamma,2}\left(u_N^{k-1} - u_N^k\right)^2 \int_0^t e^{-2\gamma(t-s)} \sup_{x\in[0,2\pi]} p_{t-s}(x) ds \\
&\le \mathrm{Lip}_{\sigma,N} \cdot \mathcal{N}_{\gamma,2}\left(u_N^{k-1} - u_N^k\right)^2 \int_0^\infty e^{-2\gamma s} \sup_{x\in[0,2\pi]} p_s(x) ds.
\end{aligned}
$$

Take the supremum over the x and t variables to obtain a convergent geometric series

$$
\mathcal{N}_{\gamma,2}(u_N^k - u_N^{k-1}) \le C^k \mathcal{N}_{\gamma,2}(u_N^1 - u_N^0),
$$

for γ large enough and C defined as

$$
C = \mathrm{Lip}_{\sigma,N} \int_0^\infty e^{-2\gamma s} \sup_{x\in[0,2\pi]} p_s(x) ds.
$$

$\square$

Theorem 3.3.2. *The mild solution to* (CAPN) *is pathwise unique.*

Proof. We will provide just an outline of the proof since it follows the same steps as the proof of Theorem 1.4.4. We will Let u_N and $\tilde{u}_N$ be two mild solutions to (CAPN). Direct computation gives us,

$$
\begin{aligned}
&\mathrm{E}\left[|u_N(t,x)-\tilde{u}_N(t,x)|^2\right] \\
&= \mathrm{E}\left[\left|\int_0^t\int_0^{2\pi} p_{t-s}(x-y)\left(\sigma(u_N(s,y)\wedge N)-\sigma(\tilde{u}_N(s,y)\wedge N)\right)\eta(ds,dy)\right|^2\right] \\
&= \mathrm{E}\left[\int_0^t\int_0^{2\pi} p_{t-s}(x-y)^2\left(\sigma(u_N(s,y)\wedge N)-\sigma(\tilde{u}_N(s,y)\wedge N)\right)^2 dyds\right] \\
&\le \mathrm{Lip}^2_{\sigma,N}\int_0^t\int_0^{2\pi} p_{t-s}(x-y)^2\,\mathrm{E}\left[(u_N(s,y)-\tilde{u}_N(s,y))^2\right]dyds \\
&\le e^{\gamma t}\mathrm{Lip}^2_{\sigma,N}\left(\sup_{s\in[0,t]}\sup_{x\in[0,2\pi]} e^{-\gamma s}\,\mathrm{E}\left[(u_N(s,y)-\tilde{u}_N(s,y))^2\right]\right)\int_0^t e^{-\gamma(t-s)}\int_0^{2\pi} p_{t-s}(x-y)^2 dyds \\
&\le e^{\gamma t}\mathrm{Lip}^2_{\sigma,N}\left(\sup_{s\in[0,t]}\sup_{x\in[0,2\pi]} e^{-\gamma s}\,\mathrm{E}\left[(u_N(s,y)-\tilde{u}_N(s,y))^2\right]\right)\int_0^\infty e^{-\gamma s}\sup_{x\in[0,2\pi]} p_s(x)dyds.
\end{aligned}
$$

Multiply previous inequality by $e^{-\gamma t}$, take supremum over the x and y variable to obtain

$$\mathcal{N}_{\gamma,2}(u_N-\tilde{u}_N)\le C\cdot\mathcal{N}_{\gamma,2}(u_N-\tilde{u}_N),$$

where

$$C=\int_0^\infty e^{-\gamma s}\sup_{x\in[0,2\pi]} p_s(x)dyds.$$

If we choose γ large enough, we get that $C<1$, which in turn implies that $\mathcal{N}_{\gamma,2}(u_N-\tilde{u}_N)=0$. Continuity of solution (which will be proved later) gives us that u_N and $\tilde{u}_N$ are indistinguishable. □

3.4 Proof of the main Lemma

The key Lemma of this section gives us a bound on the probability that the stochastic integral term in (3.2) is large. For convenience, define

$$\mathcal{N}(t,x):=\int_0^t\int_0^{2\pi} p_{t-s}(x-y)\sigma(u_N(s,y)\wedge N)\eta(s,y). \tag{3.12}$$

We would like to emphasize that T appearing throughout the rest of this section lies in the interval (0,1].

Lemma 3.4.1 (Main Lemma)**.** *For every $\epsilon>0$, we can find a constant c and $k\ge 2$ such that*

$$\sup_{u_0\in\mathcal{J}}\mathrm{P}\left(\sup_{s\in(0,T),x\in[0,2\pi]}|\mathcal{N}(s,x)|>\Delta\right)\le c\left(\frac{N^\gamma T^{(1-1/\beta-\epsilon)/2}}{\Delta}\right)^k$$

for any $\Delta>0$ and $0\le T\le 1$, where $\mathcal{J}$ is a collection of all positive continuous function on a circle.

The proof of Lemma 3.4.1 uses Kolmogorov's continuity theorem. As a byproduct, we will establish continuity *[existence of a continuous modification]* of a solution to (CAPN). We will start our proof by finding estimates for spatial difference in section 3.4.1 and time difference in section 3.4.2.

3.4.1 Space difference

We can use the Burkholder-Davis-Gundy (BDG) inequality and get

$$
\begin{aligned}
&\mathrm{E}\left[|\mathcal{N}(t,x)-\mathcal{N}(t,z)|^k\right] = \mathrm{E}\left[\left|\int_0^t\int_0^{2\pi}\left(p_{t-s}(x-y)-p_{t-s}(z-y)\right)\sigma(u_N(s,y)\wedge N)\eta(s,y)\right|^k\right] \\
&\leq (4k)^{k/2}\,\mathrm{E}\left[\left|\int_0^t\int_0^{2\pi}\left(p_{t-s}(x-y)-p_{t-s}(z-y)\right)^2\sigma(u_N(s,y)\wedge N)^2 dyds\right|^{k/2}\right] \\
&\leq (4k)^{k/2}N^{k\gamma}\left(\int_0^t\int_0^{2\pi}(p_{t-s}(x-y)-p_{t-s}(z-y))^2 dyds\right)^{k/2} \\
&\leq (4k)^{k/2}N^{k\gamma}(2\pi)^{k/2}\left(\int_0^t 2\sup_{z\in[0,2\pi]}p_{t-s}(z)\int_0^{2\pi}|p_{t-s}(x-y)-p_{t-s}(z-y)|dyds\right)^{k/2}.
\end{aligned}
$$

There are two terms we need to estimate. The supremum can be bounded by

$$
\begin{aligned}
\sup_{x\in[0,2\pi]}p_s(x) &\leq \sum_{j\in\mathbb{Z}}\sup_{x\in[0,2\pi]}\tilde{p}_s(x+2\pi j) \\
&= \sum_{j\in\mathbb{Z}\cap[-R,R]}\sup_{x\in[0,2\pi]}\tilde{p}_s(x+2\pi j) + \sum_{j\in\mathbb{Z}\cap[-R,R]^c}\sup_{x\in[0,2\pi]}\tilde{p}_s(x+2\pi j) \\
&\leq \mathrm{const}\cdot s^{-1/\beta} + \mathrm{const}\cdot\sum_{j\in\mathbb{Z}\cap[-R,R]^c}\frac{1}{(2\pi j)^{1+\alpha}} \leq \mathrm{const}\cdot s^{-1/\beta}. \qquad (3.13)
\end{aligned}
$$

And the bound on the integral of spatial difference of densities is

$$
\begin{aligned}
&\int_0^{2\pi}|p_s(x-y)-p_s(z-y)|dy \leq 2\wedge\left(\sum_{j\in\mathbb{Z}}2\pi\sup_{y\in[0,2\pi]}|\tilde{p}_s(x-y-2\pi j)-\tilde{p}_s(z-y-2\pi j)|\right) \\
&\leq 2\wedge\left(\mathrm{const}\left(L\frac{(x-z)^2}{t^{3/\beta}}+\sum_{j\in\mathbb{Z}\cap[-L,L]^c}\sup_{y\in[0,2\pi]}|\tilde{p}_s(x-y-2\pi j)-\tilde{p}_s(z-y-2\pi j)|\right)\right) \\
&\leq 2\wedge\left(\mathrm{const}\left(L\frac{(x-z)^2}{s^{3/\beta}}+\sum_{j\in\mathbb{Z}\cap[-L,L]^c}\frac{1}{j^{1+\alpha}}\right)\right) \leq 2\wedge\left(\mathrm{const}\left(L\frac{(x-z)^2}{s^{3/\beta}}+L^{-\alpha}\right)\right).
\end{aligned}
\qquad (3.14)
$$

We would like the term on the right-hand side of (3.14) to be of order $|x-z|$, which translates to $L \propto |x-y|^{-1/\alpha}$. This yields

$$\int_0^{2\pi} |p_s(x-y) - p_s(z-y)|dy \leq 2 \wedge \left(\text{const} \cdot \frac{|x-z|^{2-1/\alpha}}{s^{3/\beta}} + \text{const} \cdot |x-y| \right)$$
$$\leq \text{const} \cdot \left(1 \wedge \frac{|x-y|}{s^{3/\beta}} \right) \leq \text{const} \left(1 \wedge \frac{|x-y|}{s^3} \right),$$

and holds for $s \in [0,1]$. The almost final estimate for the spatial difference is

$$\mathrm{E}\left[|\mathcal{N}(t,x) - \mathcal{N}(t,z)|^k \right] \leq \text{const} \cdot N^{k\gamma} \left(\int_0^t \frac{1}{s^{1/\beta}} \left(1 \wedge \frac{|x-y|}{s^3} \right) ds \right)^{k/2}.$$

Use identity $r \wedge 1 \leq r^\theta$ for $\theta \in [0,1]$ and conclude

$$\mathrm{E}\left[|\mathcal{N}(t,x) - \mathcal{N}(t,z)|^k \right] \leq \text{const} \cdot N^{k\gamma} \left(|x-y|^\theta \, t^{1-1/\beta-3\theta} \right)^{k/2}. \tag{3.15}$$

3.4.2 Time difference

The first step is to write $\mathcal{N}(t+\epsilon, x)$ and $\mathcal{N}(t,x)$ from (3.12), use identity $|a+b|^k \leq 2^k |a|^k + 2^k |b|^k$ and BDG inequality to get

$$\mathrm{E}\left[|\mathcal{N}(t+\epsilon,x) - \mathcal{N}(t,x)|^k \right] \leq 2^k \mathcal{A} + 2^k \mathcal{B},$$

where

$$\mathcal{A} = (4k)^{k/2}\, \mathrm{E}\left[\left| \int_0^t \int_0^{2\pi} \left(p_{t+\epsilon-s}(x-y) - p_{t-s}(x-y) \right)^2 \sigma(u(s,y)_N \wedge N)^2 dyds \right|^{k/2} \right],$$
$$\mathcal{B} = (4k)^{k/2}\, \mathrm{E}\left[\left| \int_t^{t+\epsilon} \int_0^{2\pi} p_{t+\epsilon-s}(x-y)^2 \sigma(u_N(s,y) \wedge N)^2 dyds \right|^{k/2} \right].$$

The bound on $\mathcal{B}$ can be obtained by direct computations and with the help of the estimate on $\sup_x p_t(x)$ in (3.13), we have

$$\begin{aligned}
\mathcal{B} &\leq (4k)^{k/2}\, \mathrm{E}\left[\left| \int_t^{t+\epsilon} \int_0^{2\pi} p_{t+\epsilon-s}(x-y)^2 \sigma(u_N(s,y) \wedge N)^2 dyds \right|^{k/2} \right] \\
&\leq (4k)^{k/2} N^{\gamma k} \left(\int_0^\epsilon \int_0^{2\pi} p_s(x-y)^2 dyds \right)^{k/2} \\
&\leq \text{const} \cdot N^{\gamma k} \left(\int_0^\epsilon \sup_z p_s(z) \int_0^{2\pi} p_s(x-y) dyds \right)^{k/2} \leq \text{const} \cdot N^{\gamma k} \cdot \left(\int_0^\epsilon \sup_z p_s(z) ds \right)^{k/2} \\
&\leq \text{const} \cdot N^{\gamma k} \cdot \left(\int_0^\epsilon s^{-1/\beta} ds \right)^{k/2} \leq \text{const} \cdot N^{\gamma k} \cdot \left(\epsilon^{1-1/\beta} \right)^{k/2}.
\end{aligned}$$

We will use a similar technique for $\mathcal{A}$ and write

$$\begin{aligned}
\mathcal{A} &\leq \text{const} \cdot N^{\gamma k} \left(\int_0^t \int_0^{2\pi} \left(p_{t+\epsilon-s}(x-y) - p_{t-s}(x-y) \right)^2 dyds \right)^{k/2} \\
&\leq \text{const} \cdot N^{\gamma k} \cdot \left(\int_0^t \sup_z p_{t-s}(z) \int_0^{2\pi} \left| p_{t+\epsilon-s}(x-y) - p_{t-s}(x-y) \right| dyds \right)^{k/2}.
\end{aligned}$$

The estimate of the spatial difference of density p is also needed. Thanks to (3.10) and Preposition 3.2.3, we can write

$$\int_0^{2\pi} |p_{\epsilon+s}(x-y) - p_s(x-y)|\, dy \le \text{const}\left(1 \wedge \left(L \sup_{z\in\mathbb{R}} |\tilde{p}_{s+\epsilon}(z) - \tilde{p}_s(z)| + \sum_{j\in\mathbb{Z}\cap[-L,L]^c} \frac{1}{j^{1+\alpha}}\right)\right)$$
$$\le \text{const}\left(1 \wedge \left(L\frac{\epsilon}{s^{(1+1/\beta)}} + L^{-\alpha}\right)\right) = \text{const}\left(1 \wedge \left(\frac{\epsilon^{1-1/\alpha}}{s^{1+1/\beta}}\right)\right), \tag{3.16}$$

for a choice of $L = \epsilon^{-1/\alpha}$. The upper bound for $\mathcal{A}$ is now

$$\mathcal{A} \le \text{const} \cdot N^{\gamma k}\left(\int_0^t s^{-1/\beta}\left(1 \wedge \frac{\epsilon^{1-1/\alpha}}{s^{1+1/\beta}}\right) ds\right)^{k/2}.$$

Lastly, we use the identity $1 \wedge r \le r^\theta,\ \theta \in (0,1]$ and get

$$\mathcal{A} \le \text{const} \cdot N^{\gamma k}\left(\int_0^t s^{-1/\beta-\theta-\theta/\beta}\epsilon^{\theta-\theta/\alpha} ds\right)^{k/2} \le \text{const} \cdot N^{\gamma k}\left(t^{1-1/\beta-\theta-\theta/\beta}\epsilon^{\theta-\theta/\alpha}\right)^{k/2},$$

which yields the final estimate for the temporal difference

$$\mathrm{E}\left[|\mathcal{N}(t+\epsilon,x) - \mathcal{N}(t,x)|^k\right] \le \text{const} \cdot N^{\gamma k}\left(\left(\epsilon^{1-1/\beta}\right)^{k/2} + \left(t^{1-1/\beta-\theta-\theta/\beta}\epsilon^{\theta-\theta/\alpha}\right)^{k/2}\right)$$
$$\le \text{const} \cdot N^{\gamma k}\left(T^{1-1/\beta-\theta-\theta/\beta}\epsilon^{\theta-\theta/\alpha}\right)^{k/2}. \tag{3.17}$$

3.4.3 Use of Kolmogorov's continuity theorem

For every $(s,x),(t,y)$ from $D := [0,1] \times [0,2\pi]$, the following holds for $k \ge 2$

$$\mathrm{E}\left[|\mathcal{N}(s,x) - \mathcal{N}(t,y)|^k\right]$$
$$\le \text{const} \cdot N^{k\gamma}\left(\left(T^{1-1/\beta-3\theta}|x-y|^\theta\right)^{k/2} + \left(T^{1-1/\beta-3\theta}|t-s|^{\theta-\theta/\alpha}\right)^{k/2}\right), \tag{3.18}$$

thanks to estimates (3.15) and (3.17) in sections 3.4.1 and 3.4.2, where θ is arbitrary and $\theta \in (0,1)$. From the subadditivity of $z \mapsto z^{1/k}$, we have

$$\|\mathcal{N}(s,x) - \mathcal{N}(t,y)\|_{L^k(\mathrm{P})} \le \text{const} \cdot N^\gamma T^{(1-1/\beta-3\theta)/2}\left(|x-y|^{\theta/2} + |t-s|^{\theta(1-1/\alpha)/2}\right).$$

Let $\rho(t,x) = T^{(1-1/\beta-3\theta)/2}\left(|x-y|^{\theta/2} + |t-s|^{\theta(1-1/\alpha)/2}\right)$. Then, Kolmogorov's continuity theorem states that there is a continuous modification of $\mathcal{N}$ such that

$$\mathrm{E}\left[\sup_{(s,x),(t,y)\in D}\left|\frac{\mathcal{N}(s,x) - \mathcal{N}(t,y)}{\rho(s-t,x-y)^q}\right|^k\right] < \text{const} \cdot N^{k\gamma}, \tag{3.19}$$

for every $q \in (0, 1 - H/k)$, where $H = 2/\theta + 2/(\theta(1 - 1/\alpha))$. By Chebyshev's inequality and (3.19), we can write

$$\begin{aligned} \mathrm{P}\left\{ \sup_{(s,x),(t,y)\in D} |\mathcal{N}(s,x) - \mathcal{N}(t,y)| > \Delta \right\} &\leq \text{const} \cdot \frac{N^{k\gamma}}{\Delta^k} \rho(T, 2\pi)^{qk} \\ &\leq \text{const}(1 + (2\pi)^{aqk}) \left(\frac{N^\gamma T^{q(1-1/\beta-3\theta)/2}}{\Delta} \right)^k . \end{aligned} \tag{3.20}$$

Note that the exponent of T can be made arbitrarily close to $(1 - 1/\beta)/2$. Inequality (3.20) concludes the proof of Lemma 3.4.1.

3.5 Positivity and a comparison principle

We will need a result about the positivity of solutions to (SHE). This is because the classical result [41] for Stochastic Heat Equation is not enough. In [14], Kunwoo Kim and Le Chen established a comparison principle when $\mathcal{L}$ is the generator of an α-stable Lévy process. That too is not enough, but the proofs in [14] apply with minor changes to $\mathcal{L}$ that satisfies (A1)-(A3). We will follow proof of [14, Thm 1.1] very closely.

Theorem 3.5.1 (Weak Comparison principle)**.** *[14] Suppose that conditions (A1)-(A3) are satisfied, then the solution to* (CAPN) *is nonnegative.*

Let us define

$$\mathcal{L}^\epsilon = \frac{\mathbf{p}_\epsilon - \mathbf{I}}{\epsilon},$$

where $\mathbf{I}$ is the identity operator and

$$\mathbf{p}_t f(x) = \int_0^{2\pi} p_t(x-y) f(y) dy.$$

It is not hard to see that for

$$\mathfrak{P}_t^\epsilon := \exp(t\mathcal{L}^\epsilon) = e^{-t/\epsilon} \sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!} \mathbf{p}_{n\epsilon} = e^{-t/\epsilon}\mathbf{I} + \mathbf{R}_t^\epsilon, \tag{3.21}$$

the function $v(t,x) := \mathfrak{P}_t^\epsilon * f(x)$ solves the following initial value problem:

$$\begin{aligned} \frac{\partial}{\partial t} v &= \mathcal{L}^\epsilon v, \\ v(0,x) &= g(x). \end{aligned}$$

The operator $\mathbf{R}_t^\epsilon$ has a convolution kernel equal to

$$R_t^\epsilon = e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} p_{n\epsilon},$$

by which we precisely mean that $\mathbf{R}_t^\epsilon f(x) = \int_0^{2\pi} R_t^\epsilon(x-y)f(y)dy$. The convolution kernel for $\mathfrak{P}_t^\epsilon$ is given by

$$p_t^\epsilon(x) = e^{-t/\epsilon}\delta(x) + R_t^\epsilon(x). \tag{3.22}$$

Before we start the proof of Theorem 3.5.1, we will need a few Lemmas which all essentially state that R_t^ϵ "close" to p_t.

3.5.1 Supporting Lemmas for Theorem 3.5.1

Lemma 3.5.2. *Fix $T > 1$ and let $T > t > 0$, we have that*

$$\int_0^{2\pi} |R_t^\epsilon(x) - p_t(x)|\, dx \leq e^{-t/\epsilon} + C \cdot \frac{\epsilon^{(\alpha-1)/2\alpha}}{t^{\frac{\alpha+1}{2\alpha}-1/\beta}},$$

for some constant $C > 0$.

Proof. Let us look back at the equation (3.16). We can use it again together with Preposition 3.2.3 for $2T$ instead of T and write

$$\begin{aligned}\int_0^{2\pi} |p_{t+\epsilon}(x-y) - p_t(x-y)|\, dy &\leq \text{const}\left(1 \wedge \left(L \sup_{z\in\mathbb{R}} |\tilde{p}_{t+\epsilon}(z) - \tilde{p}_t(z)| + \sum_{j\in\mathbb{Z}\cap[-L,L]^c} \frac{1}{j^{1+\alpha}}\right)\right) \\ &\leq \text{const}\left(1 \wedge \left(L\frac{\epsilon}{t^{(1+1/\beta)}} + L^{-\alpha}\right)\right) = \text{const}\left(1 \wedge \left(\frac{\epsilon^{1-1/\alpha}}{t^{1+1/\beta}}\right)\right),\end{aligned}$$

which is valid for $\epsilon \in (-t, 2T-t)$. For $\epsilon > 2T - t$, simply bound the whole interval by 2. We can find a constant const such that

$$\int_0^{2\pi} |p_{\epsilon+t}(x-y) - p_t(x-y)|\, dy \leq \text{const}\left(1 \wedge \frac{\epsilon^{1-1/\alpha}}{t^{1+1/\beta}}\right) \leq \text{const}\frac{\epsilon^{1-1/\alpha}}{t^{1+1/\beta}} \tag{3.23}$$

for $t \in (0, T]$ and any $\epsilon \in (-t, \infty)$. Use the definition of R_t^ϵ, inequality (3.23) and Hölder's inequality to write

$$\begin{aligned}&\int_0^{2\pi} |R_t^\epsilon(x) - p_t(x)|\, dx \leq e^{-t/\epsilon} + e^{-t/\epsilon}\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}\int_0^{2\pi} |p_{n\epsilon}(x) - p_t(x)|\, dx \\ &\leq e^{-t/\epsilon} + \text{const}\cdot e^{-t/\epsilon}\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}\frac{(n\epsilon - t)^{1-1/\alpha}}{t^{1+1/\beta}} \leq e^{-t/\epsilon} + \text{const}\cdot\frac{e^{-t/\epsilon}}{t^{1+1/\beta}}\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}(n\epsilon - t)^{1-1/\alpha} \\ &\leq e^{-t/\epsilon} + \frac{e^{-t/\epsilon}}{t^{1+1/\beta}}\left(\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}\right)^{(\alpha+1)/2\alpha}\left(\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}(\epsilon n - t)^2\right)^{(\alpha-1)/2\alpha}. \end{aligned} \tag{3.24}$$

For simplicity, we can define

$$z = t/\epsilon \text{ and } P_k^n := k(k-1)\cdots(k-n+1).$$

It is not hard to see that $\sum_{k=1}^{\infty} \frac{z^k}{k!} P_k^n = e^z z^n$. Since $k^2 = P_k^2 + P_k^1$, we can rewrite the second term of (3.24) as

$$\begin{aligned}
&\frac{1}{t^{1+1/\beta}} e^{-z(\alpha-1)/2\alpha} \left(\epsilon^2 \sum_{n=1}^{\infty} \frac{z^n}{n!} (n^2 - 2nz + z^2) \right)^{(\alpha-1)/2\alpha} \\
&\leq \frac{1}{t^{1+1/\beta}} e^{-z(\alpha-1)/2\alpha} \left(\epsilon^2 e^z (z^2 + z - 2z^2 + z^2) - \epsilon^2 z^2 \right)^{(\alpha-1)/2\alpha} \\
&\leq \frac{1}{t^{1+1/\beta}} e^{-z(\alpha-1)/2\alpha} \left(\epsilon^2 e^z z \right)^{(\alpha-1)/2\alpha}, \qquad (3.25)
\end{aligned}$$

and if we switch back to original coordinates (ϵ and t), we get

$$\frac{1}{t^{1+1/\beta}} \left(\epsilon^2 \frac{t}{\epsilon} \right)^{(\alpha-1)/2\alpha} \leq \frac{\epsilon^{(\alpha-1)/2\alpha}}{t^{\frac{\alpha+1}{2\alpha} - 1/\beta}}.$$

This concludes the proof of the Lemma. □

Lemma 3.5.3. *For $t > s > 0$, we have*

$$\sup_x p_t(x) \leq \sup_x p_s(x).$$

Proof. This is an easy consequence of semigroup properties of p. Let $t = s + \delta$ where $\delta > 0$, then we have

$$p_t(x) = \int_0^{2\pi} p_\delta(x-y) p_s(y) dy \leq \sup_x p_s(x) \int_0^{2\pi} p_\delta(x-y) dy = \sup_x p_s(x).$$

Taking supremum over x concludes the Lemma. □

Lemma 3.5.4. *Fix T, for $T \geq t > 0$, we have*

$$\int_0^{2\pi} R_t^\epsilon(x)^2 dx \leq \text{const} \left(t^{-1/\beta} + 1 \right).$$

Proof. Use Tonelli's theorem and definition of R_t^ϵ to write

$$\begin{aligned}
&\int_0^{2\pi} R_t^\epsilon(x)^2 dx = e^{-2t/\epsilon} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{(t/\epsilon)^n}{n!} p_{n\epsilon}(x) \sum_{m=1}^{\infty} \frac{(t/\epsilon)^m}{m!} p_{m\epsilon}(x) dx \\
&= e^{-2t/\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(t/\epsilon)^{n+m}}{n! m!} \int_0^{2\pi} p_{n\epsilon}(x) p_{m\epsilon}(x) dx \\
&\leq e^{-2t/\epsilon} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(t/\epsilon)^{n+m}}{n! m!} \sup_x p_{n\epsilon}(x) \int_0^{2\pi} p_{m\epsilon}(x) dx \leq e^{-t/\epsilon} \sum_{n=1}^{\infty} \frac{(t/\epsilon)^n}{n!} \sup_x p_{n\epsilon}(x).
\end{aligned}$$

We also have bound $\sup_x p_{n\epsilon} \leq \text{const} \left((\epsilon n)^{-1/\beta} + 1 \right)$, which is a consequence of Lemma 3.5.3 and identity (3.13). So far, we have

$$\int_0^{2\pi} R_t^\epsilon(x)^2 dx \leq \text{const} \cdot e^{-t/\epsilon} \sum_{n=1}^{\infty} \frac{(t/\epsilon)^n}{n!} (\epsilon n)^{-1/\beta} + \text{const}.$$

Use Hölder's inequality on the first term to obtain

$$
\begin{aligned}
e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}(\epsilon n)^{-1/\beta} &\le e^{-t/\epsilon}\left(\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}\right)^{(\beta-1)/\beta}\left(\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}(\epsilon n)^{-1}\right)^{1/\beta}\\
&= e^{-t/(\beta\epsilon)}t^{-1/\beta}\left(\sum_{n=1}^{\infty}\frac{(t/\epsilon)^{n+1}}{(n+1)!}\frac{n+1}{n}\right)^{1/\beta} \le e^{-t/(\beta\epsilon)}t^{-1/\beta}\left(2\sum_{n=1}^{\infty}\frac{(t/\epsilon)^{n+1}}{(n+1)!}\right)^{1/\beta}\\
&\le t^{-1/\beta}\left(2-2e^{-t/\epsilon}-2(t/\epsilon)e^{-t/\epsilon}\right)^{1/\beta} \le 2\cdot t^{-1/\beta}, \qquad (3.26)
\end{aligned}
$$

where the last step is due to maximization over $z = t/\epsilon \ge 0$. This completes the proof. □

Lemma 3.5.5. *Fix $T > 0$, there exist δ and constant C such that for $T \ge t > 0$ and $1 > \epsilon > 0$*

$$\int_0^t \int_0^{2\pi} (R_s^\epsilon(z) - p_s(z))^2\, dzds \le C\epsilon^\delta.$$

Proof. Use the definition of R_s^ϵ together with Jensen's inequality with respect to the probability measure $e^{-s/\epsilon}\frac{(s/\epsilon)^n}{n!}$ on $\mathbb{N}\cup\{0\}$. Note that the first term corresponding to $n = 0$ is taken to be zero. Thus, we obtain

$$
\begin{aligned}
I &= \int_0^t\int_0^{2\pi}(R_s^\epsilon(z)-p_s(z))^2\,dzds = \int_0^t\int_0^{2\pi}\left(e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}(p_{n\epsilon}(z)-p_s(z))\right)^2 dzds\\
&\le \int_0^t\int_0^{2\pi} e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}(p_{n\epsilon}(z)-p_s(z))^2 dzds\\
&\le \int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}\int_0^{2\pi}(p_{n\epsilon}(z)-p_s(z))^2 dzds\\
&\le \int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}\left(\sup_x p_s(x)+\sup_x p_{n\epsilon}(x)\right)\int_0^{2\pi}|p_{n\epsilon}(z)-p_s(z)|\,dzds \le \mathrm{const}(I_1+I_2).
\end{aligned}
$$

To bound I_1, we will again use that $\sup_x p_{n\epsilon} \le \mathrm{const}\left((\epsilon n)^{-1/\beta}+1\right)$. This is a consequence of Lemma 3.5.3 and identity (3.13). The bound on I_1 is

$$
\begin{aligned}
I_1 &= \int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}\sup_x p_{n\epsilon}(x)\int_0^{2\pi}|p_{n\epsilon}(z)-p_s(z)|\,dzds\\
&\le \mathrm{const}\int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}\left((n\epsilon)^{-1/\beta}+1\right)\int_0^{2\pi}|p_{n\epsilon}(z)-p_s(z)|\,dzds \le \mathrm{const}(I_3+I_4)
\end{aligned}
$$

where I_3 is

$$
\begin{aligned}
I_3 &= \int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}(n\epsilon)^{-1/\beta}\int_0^{2\pi}|p_{n\epsilon}(z)-p_s(z)|\,dzds\\
&\le \mathrm{const}\int_0^t e^{-s/\epsilon}\sum_{n=1}^{\infty}\frac{(s/\epsilon)^n}{n!}(n\epsilon)^{-1/\beta}\left(1\wedge\frac{|n\epsilon-s|^{1-1/\alpha}}{s^{1+1/\beta}}\right)dzds,
\end{aligned}
$$

and the last step is due to (3.23).

Use the fact that $1 \wedge r \le r^\theta$ and Hölder's inequality to further write

$$I_3 \le \text{const} \int_0^t \frac{e^{-s/\epsilon}}{s^{\theta(\beta+1)/\beta}} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} (n\epsilon)^{-1/\beta} |n\epsilon - s|^{\theta(\alpha-1)/\alpha}\, dzds$$
$$\le \text{const} \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} \left(e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} (n\epsilon)^{-1} \right)^{1/\beta}$$
$$\cdot \left(e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} |n\epsilon - s|^{\theta(\alpha-1)/\alpha} \right)^{(\beta-1)/\beta} dzds.$$

Another use of Jensen's inequality with respect to probability measure $e^{-s/\epsilon}\frac{(s/\epsilon)^n}{n!}$ on $\mathbb{N}\cup\{0\}$ (with zero term for $n=0$) gives us

$$I_3 \le \text{const} \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} \left(e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} (n\epsilon)^{-1} \right)^{1/\beta} \cdot$$
$$\left(e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} (n\epsilon - s)^2 \right)^{\frac{\theta(\alpha+1)(\beta-1)}{2\alpha\beta}} dzds.$$

Previous estimates (3.26) and both (3.24),(3.25) tell us that

$$I_3 \le \text{const} \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} s^{-1/\beta} (\epsilon s)^{\frac{\theta(\alpha+1)(\beta-1)}{2\alpha\beta}} ds. \tag{3.27}$$

Therefore, we can find $\delta_1 > 0$ (and corresponding θ) such that $I_3 \le \text{const} \cdot \epsilon^{\delta_1}$. The estimate on I_4 will go the similar way as the estimate for I_3. Write

$$I_4 = \int_0^t e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} \int_0^{2\pi} |p_{n\epsilon}(z) - p_s(z)|\, dzds$$
$$\le \int_0^t e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} \left(1 \wedge \frac{|n\epsilon - s|^{1-1/\alpha}}{s^{1+1/\beta}} \right) ds$$
$$\le \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} |n\epsilon - s|^{\theta(\alpha-1)/\alpha}\, ds$$
$$\le \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} \left(e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} (n\epsilon - s)^2 \right)^{\theta(\alpha-1)/2\alpha} ds$$
$$\le \text{const} \int_0^t \frac{1}{s^{\theta(\beta+1)/\beta}} (\epsilon s)^{\theta(\alpha-1)/2\alpha} ds \le \text{const} \cdot \delta_2$$

for some $\delta_2 > 0$ and corresponding θ. Because we have $s \in (0, T)$, we can simply use identity (3.13) to write

$$I_2 = \int_0^t e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} \sup_x p_s(x) \int_0^{2\pi} |p_{n\epsilon}(z) - p_s(z)|\, dzds$$
$$\le \text{const} \int_0^t s^{-1/\beta} e^{-s/\epsilon} \sum_{n=1}^{\infty} \frac{(s/\epsilon)^n}{n!} \int_0^{2\pi} |p_{n\epsilon}(z) - p_s(z)|\, dzds.$$

We can take identical steps as in the estimate for I_4 to get that there exists δ_3 such that $I_2 \leq \text{const} \cdot \epsilon^{\delta_3}$. Overall, we have that $I \leq \text{const}(I_3 + I_4 + I_2)$. Take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ to finish the proof. $\square$

Lemma 3.5.6. *Let $\mathfrak{v}_\epsilon$ be defined as in* (3.31), *then*

$$\lim_{\epsilon \to 0} \epsilon \mathfrak{v}_\epsilon = 0.$$

Proof. Direct computation gives us

$$\epsilon \int_0^{2\pi} \phi_\epsilon^2(y)dy \leq \epsilon \cdot \sup_x \phi_\epsilon(x) \int_0^{2\pi} \phi_\epsilon(y)dy = \epsilon \cdot \sup_x \phi_\epsilon(x).$$

For $\epsilon \in (0,1)$, we can write

$$\epsilon \cdot \sup_x \phi_\epsilon(x) \leq \text{const} \frac{\epsilon}{\epsilon^{1/2}} \leq \text{const} \cdot \epsilon^{1/2},$$

by a similar argument as in (3.13). $\square$

3.5.2 Proof of Theorem 3.5.1

Take $a_n = -2(n^2 + n + 2)^{-1}$ for $n \in \mathbb{N}$, then $\int_{a_{n-1}}^{a_n} x^{-2}dx = n$. Also let $\psi_n, n \in \mathbb{N}$ be a function such that it is supported on (a_{n-1}, a_n) and

$$0 \leq \psi_n(x) \leq \frac{2}{nx^2} \quad \text{and} \quad \int_{a_{n-1}}^{a_n} \psi_n(x)dx = 1. \tag{3.28}$$

Also define Ψ as

$$\Psi_n(x) := \int_0^x \int_0^y \psi_n(z)dzdy.$$

The following is true:

- $\Psi_n \in C^2(\mathbb{R}),\ \Psi_n''(x) = \psi(x)$
- $-1 \leq \Psi_n'(x) \leq 0$
- For every $x \in \mathbb{R}$, we have $\Psi_n(x) \uparrow -(x \wedge 0) =: \Psi(x),\ \Psi_n'(x) \downarrow -\mathbb{1}_{(-\infty,0)}(x)$ and $\Psi_n'(x)x \uparrow \Psi(x)$.

Functions Ψ_n will play a role in Itô's formula, where the argument of Ψ_n will be perturbed solution u_N. To make this happen, we will also need to smooth the noise in spatial variable. Define

$$\phi_\epsilon(x) = \sum_{j\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(x+j2\pi)^2}{2\epsilon}\right), \tag{3.29}$$

and

$$\eta_x^\epsilon([0,t]) = \int_0^t \int_0^{2\pi} \phi_\epsilon(x-y)\eta(ds,dy).$$

The perturbed solution u_N, called u_N^ϵ, will be a solution to

$$\begin{aligned} \frac{\partial}{\partial t} u_N^\epsilon &= \mathcal{L}^\epsilon u_N^\epsilon + \sigma(u_N^\epsilon \wedge N)\eta_x^\epsilon(dt) \\ u_N^\epsilon(0,x) &= g(x). \end{aligned} \tag{3.30}$$

In the first part of the proof, we will show that u_N^ϵ is positive. In the second part, we will show that u_N^ϵ is close to u_N in $L^2(P)$ norm. Noise η_x^ϵ can be viewed as a Brownian motion for every x. The quadratic variation of $X_t := \eta_x^\epsilon([0,t])$ for some fixed x will be, thanks to Isometry 1.4.3,

$$\langle X\rangle_t = t\mathfrak{v}_\epsilon \text{ where } \mathfrak{v}_\epsilon := \int_0^{2\pi} \phi_\epsilon^2(y)dy. \tag{3.31}$$

The perturbed solution $u_N^\epsilon(t,x)$ can be written as a strong solution

$$u_N^\epsilon(t,x) = u_N^\epsilon(0,x) + \int_0^t \mathcal{L}^\epsilon u_N^\epsilon(s,x)ds + \int_0^t \sigma(u_N^\epsilon(s,x)\wedge N)\eta_x^\epsilon(ds), \tag{3.32}$$

due to Theorem 3.5.7. The Itô's formula gives us

$$\begin{aligned} \Psi_n(u_N^\epsilon(t,x)) = &\int_0^t \Psi_n'(u_N^\epsilon(t,x))\sigma(u_N^\epsilon(s,x))\eta_x^\epsilon(ds) + \frac{1}{2}\int_0^t \Psi_n''(u_N^\epsilon(s,x))\sigma(u_N^\epsilon(s,x))^2\mathfrak{v}_\epsilon ds \\ &+ \int_0^t \Psi_n'(u_N^\epsilon(s,x))\mathcal{L}^\epsilon u_N^\epsilon(s,x)ds. \end{aligned} \tag{3.33}$$

Let $\mathrm{Lip}_{\sigma N}$ be the Lipschitz constant of a function $\sigma(\cdot\wedge N)$. Use (3.28) to further estimate

$$\begin{aligned} \frac{1}{2}\int_0^t \Psi_n''(u_N^\epsilon(s,x))\sigma(u_N^\epsilon(s,x))^2\mathfrak{v}_\epsilon ds &= \frac{1}{2}\int_0^t \Psi_n''(u_N^\epsilon(s,x))\left(\sigma(u_N^\epsilon(s,x)) - \sigma(0)\right)^2 \mathfrak{v}_\epsilon ds \\ &\le \frac{\mathrm{Lip}_{\sigma N}^2\ t}{n}\mathfrak{v}_\epsilon. \end{aligned}$$

If we take the expectation of (3.33), we arrive at

$$\mathrm{E}\left[\Psi_n(u_N^\epsilon(t,x))\right] \le \frac{\mathrm{Lip}_{\sigma,N}^2\ t}{n}\mathfrak{v}_\epsilon + \mathrm{E}\left[\frac{1}{\epsilon}\int_0^t \Psi_n'(u_N^\epsilon(s,x))\int_0^{2\pi} p_\epsilon(x-y)\left(u_N^\epsilon(s,y) - u_N^\epsilon(s,x)\right)dyds\right].$$

Naturally, we would like to take a limit as $n \uparrow \infty$. Monotone convergence theorem gives us

$$
\begin{aligned}
&\mathrm{E}\left[\Psi(u_N^\epsilon(t,x))\right] \\
&\quad \le \frac{1}{\epsilon}\int_0^t \mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,y)<0\right)u_N^\epsilon(s,y)\right]ds \\
&\qquad - \frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,x)<0\right)u_N^\epsilon(s,y)\right]dyds.
\end{aligned}
$$

The second term can we rewritten as

$$
\begin{aligned}
&-\frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,x)<0\right)u_N^\epsilon(s,y)\right]dyds \\
&\quad \le -\frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,x)<0, u_N^\epsilon(s,y)<0\right)u_N^\epsilon(s,y)\right]dyds \\
&\quad \le \frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,x)<0, u_N^\epsilon(s,y)<0\right)|u_N^\epsilon(s,y)|\right]dyds \\
&\quad \le \frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,y)<0\right)|u_N^\epsilon(s,y)|\right]dyds \\
&\quad \le \frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\Psi(u_N^\epsilon(s,y))\right]dyds.
\end{aligned}
$$

Note that $\int_0^t \mathrm{E}\left[\mathbb{1}\left(u_N^\epsilon(s,y)<0\right)u_N^\epsilon(s,y)\right]ds \le 0$, which finally gives us

$$
\mathrm{E}\left[\Psi(u_N^\epsilon(t,x))\right] \le \frac{1}{\epsilon}\int_0^t\int_0^{2\pi} p_\epsilon(x-y)\,\mathrm{E}\left[\Psi(u_N^\epsilon(s,y))\right]dyds,
$$

and

$$
\sup_x \mathrm{E}\left[\Psi(u_N^\epsilon(t,x))\right] \le \frac{1}{\epsilon}\int_0^t \sup_x \mathrm{E}\left[\Psi(u_N^\epsilon(s,x))\right]ds.
$$

Gronwall's Lemma on page 89 tells us that $\mathrm{E}\left[\Psi(u_N^\epsilon(s,x))\right] = 0$ for every $t > 0$ and $x \in [0, 2\pi]$. From the definition of Ψ, we get that $u_N^\epsilon(t,x) \ge 0$ almost surely for every $t > 0$ and $x \in [0, 2\pi]$.

Positivity of u_N^ϵ was established; now we will show that for every $t > 0$ and $x \in [0, 2\pi]$

$$
\lim_{\epsilon \downarrow 0} \|u_N^\epsilon(t,x) - u_N(t,x)\|_2^2 = 0,
$$

which would immediately give us positivity of u_N. The existence of strong solution u_N^ϵ of course implies the existence of the mild solution; see Theorem 3.5.8 for details. The mild solution for u_N^ϵ can be written as

$$
\begin{aligned}
u_N^\epsilon(t,x) &= (u_0 * \mathfrak{P}_t^\epsilon)(x) + \int_0^t e^{-(t-s)/\epsilon}\sigma(u_N^\epsilon(s,x)\wedge N)\eta_x^\epsilon(ds) \\
&\quad + \int_0^t \int_0^{2\pi} R_{t-s}^\epsilon(x-y)\sigma(u_N^\epsilon(s,x)\wedge N)\eta_y^\epsilon(ds)dy \\
&= (u_0 * \mathfrak{P}_t^\epsilon)(x) + \int_0^t e^{-(t-s)/\epsilon}\sigma(u_N^\epsilon(s,x)\wedge N)\eta_x^\epsilon(ds) \\
&\quad + \int_0^t \int_0^{2\pi} \left(\int_0^{2\pi} \phi_\epsilon(y-z)R_{t-s}^\epsilon(x-z)\sigma(u_N^\epsilon(s,z)\wedge N)dz\right)\, \eta(ds,dy). \qquad (3.34)
\end{aligned}
$$

The transition from the first equality to the second one, more precisely the change of integral with respect to $\eta_y^\epsilon(ds)dy$ to integral with respect to $\eta(ds,dy)$, can be easily justified by computing the $L^2(P)$ norm of the difference of both integrals. This is yet another example of the Stochastic Fubini theorem. This theorem [version not suitable for this setting] can be found in [61, Thm. 2.6, p. 296] or in Appendix B. Write down the mild solution for u_N and (3.34) to get

$$
\begin{aligned}
&\mathrm{E}\left[(u_N^\epsilon(t,x)-u(t,x))^2\right] \\
&\leq 6\left((u_0 * \mathfrak{P}_t^\epsilon)(x) - (u_0 * p_t)(x)\right)^2 + 6\int_0^t e^{-2(t-s)/\epsilon}\mathfrak{v}_\epsilon \sigma(u_N^\epsilon(s,x)\wedge N)^2 ds \\
&+ 6\int_0^t\int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z)R_{t-s}^\epsilon(x-z)\left(\sigma(u_N^\epsilon(s,z)\wedge N)-\sigma(u_N(s,z)\wedge N)\right)dz\right)^2\right] dyds \\
&+ 6\int_0^t\int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z)R_{t-s}^\epsilon(x-z)\left(\sigma(u_N(s,z)\wedge N)-\sigma(u_N(s,y)\wedge N)\right)dz\right)^2\right] dyds \\
&+ 6\int_0^t\int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z)\left(R_{t-s}^\epsilon(x-z)-p_{t-s}(x-z)\right)\sigma(u_N(s,y)\wedge N)dz\right)^2\right] dyds \\
&+ 6\int_0^t\int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z)\left(p_{t-s}(x-z)-p_{t-s}(x-y)\right)\sigma(u_N(s,y)\wedge N)dz\right)^2\right] dyds \\
&= 6\sum_{i=1}^6 I_i,
\end{aligned}
$$

by multiple use of triangle inequality.

We will show that each I_i goes to zero as $\epsilon \downarrow 0$, with help of Lemmas in section 3.5.1. The first term can be bounded by

$$
\begin{aligned}
I_1 = (u_0 * (p_t^\epsilon - p_t))(x)^2 &\leq \left(\sup_x u_0(x)\right)^2 \left(\int_0^{2\pi}\left(e^{-t/s}\sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!}\left(p_{n\epsilon}(x)-p_t(x)\right)\right)dx\right)^2 \\
&\leq \left(\sup_x u_0(x)\right)^2 \left(\int_0^{2\pi} |p_t^\epsilon(x)-p_t(x)|\,dx\right)^2.
\end{aligned}
$$

We can use the definition (3.22) together with Lemma 3.5.2 to get

$$\lim_{\epsilon \to 0} I_1 = 0.$$

The estimate for I_2 will be

$$I_2 \le N^{2\gamma} \mathfrak{v}_\epsilon \int_0^t e^{-2(t-s)/\epsilon} ds \le \frac{N^{2\gamma}}{2} \epsilon \mathfrak{v}_\epsilon,$$

which converges to zero as $\epsilon \to 0$ due to Lemma 3.5.6. To estimate I_3, we can use Jensen's inequality with respect to the probability measure ϕ_ϵ

$$\begin{aligned} I_3 &= \int_0^t \int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z) R^\epsilon_{t-s}(x-z) \left(\sigma(u^\epsilon_N(s,z)) - \sigma(u_N(s,z)) \, dz \right)^2 \right] dyds \\ &\le \mathrm{Lip}^2_{\sigma,N} \int_0^t \int_0^{2\pi} \int_0^{2\pi} \phi_\epsilon(y-z) R^\epsilon_{t-s}(x-z)^2 \left\| u^\epsilon_N(s,z) - u_N(s,z) \right\|^2_{L^2(P)} dzdyds \\ &= \mathrm{Lip}^2_{\sigma,N} \int_0^t \int_0^{2\pi} R^\epsilon_{t-s}(x-z)^2 \left\| u^\epsilon_N(s,z) - u_N(s,z) \right\|^2_{L^2(P)} dzds, \end{aligned}$$

where $\mathrm{Lip}_{\sigma,N}$ stands for Lipschitz constant for a function $\sigma(\cdot \wedge N)$.
Define

$$f_\theta(\epsilon) = \int_0^{2\pi} |x|^\theta \, \phi_\epsilon(x) dx.$$

and use the same steps as in the previous estimate together with (3.18) to write

$$\begin{aligned} I_4 &= \int_0^t \int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z) R^\epsilon_{t-s}(x-z) \left(\sigma(u_N(s,z) \wedge N) - \sigma(u_N(s,y) \wedge N) \, dz \right)^2 \right] dyds \\ &\le \mathrm{Lip}^2_{\sigma,N} \int_0^t \int_0^{2\pi} \int_0^{2\pi} R^\epsilon_{t-s}(x-z)^2 \left\| u_N(s,z) - u_N(s,y) \right\|^2_{L^2(P)} \phi_\epsilon(z-y) dydzds \\ &\le \mathrm{const} \int_0^t \int_0^{2\pi} R^\epsilon_{t-s}(x-z)^2 \, |z-y|^\theta \, \phi_\epsilon(z-y) dydzds \\ &\le \mathrm{const} \cdot f_\theta(\epsilon) \int_0^t \int_0^{2\pi} R^\epsilon_{t-s}(x-z)^2 dzds \\ &\le \mathrm{const} \cdot f_\theta(\epsilon) \int_0^t (1 + s^{-1/\beta}) ds. \end{aligned}$$

We used Lemma 3.5.4 for $\theta \in (0,1)$ in the last step. One can see that $\lim_{\epsilon \to 0} f_\theta(\epsilon) = 0$ and therefore $\lim_{\epsilon \to 0} I_4 = 0$. Term I_5 will be

$$\begin{aligned} I_5 &= \int_0^t \int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z) \left(R^\epsilon_{t-s}(x-z) - p_{t-s}(x-z) \right) \sigma(u_N(s,y) \wedge N) dz \right)^2 \right] dyds \\ &\le N^{2\gamma} \int_0^t \int_0^{2\pi} \left(R^\epsilon_{t-s}(x-z) - p_{t-s}(x-z) \right)^2 dzds, \end{aligned}$$

which converges to zero as $\epsilon \to 0$ due to Lemma 3.5.5. Since estimate of the last term uses the above seen techniques, we write

$$\begin{aligned}
I_6 &= \int_0^t \int_0^{2\pi} \mathrm{E}\left[\left(\int_0^{2\pi} \phi_\epsilon(y-z)\left(p_{t-s}(x-z)-p_{t-s}(x-y)\right)\sigma(u_N(s,y)\wedge N)dz\right)^2\right] dyds \\
&\le N^{2\gamma} \int_0^t \int_0^{2\pi} \int_0^{2\pi} \phi_\epsilon(y-z)\left(p_s(x-z)-p_s(x-y)\right)^2 dydzds \\
&\le \text{const} \int_0^t \sup_x p_s(x) \int_0^{2\pi} \int_0^{2\pi} \phi_\epsilon(y-z)\left|p_s(x-z)-p_s(x-y)\right| dydzds \\
&\le \text{const} \int_0^t \frac{1}{s^{1/\beta}} \int_0^{2\pi} \int_0^{2\pi} \phi_\epsilon(y-z)\left(1 \wedge \frac{|z-y|}{s^3}\right) dydzds \\
&\le \text{const} \int_0^t \frac{1}{s^{1/\beta}} \int_0^{2\pi} \int_0^{2\pi} \phi_\epsilon(y-z)\frac{|z-y|^\theta}{s^{3\theta}} dydzds \le \text{const} \cdot f_\theta(\epsilon) \int_0^t \frac{1}{s^{1/\beta+3\theta}} ds.
\end{aligned}$$

Term I_6 also converges to zero with ϵ. Put all estimates for I_i together and get

$$\begin{aligned}
&\mathrm{E}\left[\left(u_N^\epsilon(t,x)-u(t,x)\right)^2\right] \\
&\qquad \le \text{const} \int_0^t \int_0^{2\pi} R_{t-s}^\epsilon(x-z)^2\, \mathrm{E}\left[\left(u_N^\epsilon(s,z)-u_N(s,z)\right)^2\right] dzds + F(\epsilon) \qquad (3.35)
\end{aligned}$$

where F is some function bounding $I_1+I_2+I_4+I_5+I_6$, that depends on ϵ and our choice of T. Of course, we have $\lim_{\epsilon\to 0} F(\epsilon) = 0$. Take supremum over the spatial variable on the left-hand side of the previous inequality, which yields

$$M(t,\epsilon) \le \text{const} \int_0^t M(s,\epsilon) \int_0^{2\pi} R_{t-s}^\epsilon(x-z)^2 dzds + F(\epsilon),$$

where

$$M(t,\epsilon) = \sup_x \mathrm{E}\left[\left(u_N^\epsilon(t,x)-u(t,x)\right)^2\right].$$

By Lemma 3.5.4, we get, for $t \in (0,T)$:

$$M(t,\epsilon) \le \text{const} \int_0^t M(s,\epsilon)(t-s)^{-1/\beta} ds + F(\epsilon). \qquad (3.36)$$

Apply Gronwall's Lemma on (3.36) and get that for every $t > 0$ and $x \in \mathbb{R}$, we have

$$\lim_{\epsilon\to 0} \|u_N^\epsilon(t,x) - u_N(t,x)\|_2^2 = 0.$$

Solution $u_N(t,x)$ is almost surely positive for every $t > 0$ and $x \in \mathbb{R}$. From continuity of u_N, we have that $u_N(t,x)$ is positive for every $t > 0$ and $x \in \mathbb{R}$ almost surely. This proves Theorem 3.5.1.

3.5.3 Strong and mild solution u_N^ϵ

Theorem 3.5.7 (Strong solution). *There exists a stochastic process u_N^ϵ which satisfies* (3.32) *for $t \geq 0$ and $x \in [0, 2\pi]$. This process is often called a strong solution to* (3.30).

Sketch of the proof. One can proceed just as in the case of Stochastic Differential Equations. Write Picard's iterations

$$u_N^{\epsilon,(1)}(t,x) = f(x) + \int_0^t \mathcal{L}^\epsilon f(x) ds + \int_0^t \sigma(f(x) \wedge N) \eta_x^\epsilon(ds)$$
$$u_N^{\epsilon,(n+1)}(t,x) = u_N^{\epsilon,(n)}(0,x) + \int_0^t \mathcal{L}^\epsilon u_N^{\epsilon,(n)}(s,x) ds + \int_0^t \sigma(u_N^{\epsilon,(n)}(s,x) \wedge N) \eta_x^\epsilon(ds).$$

Establish $L^2(P)$ convergence by letting $n \uparrow \infty$. □

Theorem 3.5.8 (Mild Solution). *Let u_N^ϵ be a strong solution to* (3.30)*, then it is also a mild solution.*

Sketch of the proof. If we adapt [18, Lem. 5.5] to our setting, we get that for any 2π-periodic continuously differentiable function ζ on $[0,T] \times \mathbb{R}$:

$$\int_0^{2\pi} u_N^\epsilon(t,x)\zeta(t,x)dx = \int_0^{2\pi} f(x)\zeta(0,x)dx + \int_0^t \int_0^{2\pi} u_N^\epsilon(s,x) \left(\frac{\partial}{\partial t}\zeta(s,x) + \mathcal{L}^{\epsilon *}\zeta(s,x) \right) dxds + \int_0^t \int_0^{2\pi} \sigma(u_N^{\epsilon,(n)}(s,x) \wedge N)\zeta(s,x)\eta_x^\epsilon(ds)dx.$$

Operator $\mathcal{L}^{\epsilon *}$ stands for $L^2([0,2\pi])$ adjoint of $\mathcal{L}^\epsilon$. Apply the above line with $\zeta(s,x) = \mathfrak{P}_{t-s}(x-z)$ and get the form of the mild solution:

$$u_N^\epsilon(t,z) = (\mathfrak{P}_t * f)(z) + \int_0^t \int_0^{2\pi} \mathfrak{P}_{t-s}(x-z)\sigma(u_N^\epsilon(s,x) \wedge N)\eta_x(ds)dy.$$

□

3.6 Additional supporting Lemmas

In this section, we will prove the following Lemma for $U(t)$, which is defined to be

$$U(t) = \int_0^{2\pi} u(t,x)dx. \tag{3.37}$$

Lemma 3.6.1. *$U(t)$ is a local supermartingale up to time $\mathfrak{t}$, where*

$$\mathfrak{t} := \lim_{N \to \infty} \mathfrak{t}_N. \tag{3.38}$$

Proof. Similarly to [40] we define

$$V_N(t) = \int_0^{2\pi} u_N(t,x)dx. \tag{3.39}$$

Both solutions u and u_N agree up to the time $\mathfrak{t}_N$, the time when the solution u_N reaches level N for the first time. This also implies that both U and V agree up to the stopping time $\mathfrak{t}_N$. Our definition of $\mathfrak{t}$ makes sense pointwise, since $\mathfrak{t}_n$ is almost surely increasing. We will show that $V_N(t)$ is a supermartingale and conclude that $U(t)$ must be a local supermartingale. The formulation (3.2) is equivalent to

$$\begin{aligned} u_N(t+t_0,x) = &\int_0^{2\pi} p_t(x-y)u_N(t_0,y)dy \\ &+ \int_0^t \int_0^{2\pi} p_{t-s}(x-y)\sigma(u_N(t_0+s,y)\wedge N)\eta(dy,ds) \ , \end{aligned} \tag{3.40}$$

thanks to the Markov property of solutions. Let us denote the second term by

$$\mathcal{N}(t_0,t,x) = \int_0^t \int_0^{2\pi} p_{t-s}(x-y)\sigma(u_N(t_0+s,y)\wedge N)\eta(dy,ds+t_0).$$

To show that V is a supermartinagale, we will need to estimate integrals of both terms in (3.40), but let us state some supporting claims first. We know that

$$\mathrm{E}\left[\mathcal{N}(t_0,t,x)|\mathscr{F}_{t_0}\right] = 0 \ , \tag{3.41}$$

where $\mathscr{F}_{t_0} = \sigma\left\{\int_0^t\int_0^{2\pi} \mathbb{1}_A(x)\eta(dx,ds) : A \in \mathscr{B}_b(\mathbb{R}) \ , \ 0 \le t \le t_0\right\}$. Observe that $L^1(\mathbb{R})$ norm of $\int_0^{2\pi} p_t(x-y)u_N(t_0,y)dy$ is going to be

$$\int_0^{2\pi}\int_0^{2\pi} p_t(x-y)u_N(t_0,y)dydx \le \sup_{z\in[0,2\pi]} \|p_t(z-\cdot)\|_{L^1(\mathbb{R})} \|v_{t_0}\|_{L^1(\mathbb{R})} \ , \tag{3.42}$$

and the term $\sup_{y\in[0,2\pi]} \|p_t(z-\cdot)\|_{L^1(\mathbb{R})}$ is equal to one for $t > 0$. Next, we will show that

$$\mathrm{E}\left[\int_0^{2\pi} \mathcal{N}(t_0,t,x)dx \middle| \mathscr{F}_{t_0}\right] = 0,$$

which is true if we can integrate (3.41) and put the integral inside the conditional expectation. In other words, we would like to show that

$$\mathrm{E}\left[\int_0^{2\pi} \mathcal{N}(t_0,t,x)dx \middle| \mathscr{F}_{t_0}\right] = \int_0^{2\pi} \mathrm{E}\left[\mathcal{N}(t_0,t,x)|\, \mathscr{F}_{t_0}\right]dx = 0. \tag{3.43}$$

Since $\mathcal{N}$ can to be taken (modification of $\mathcal{N}$) continuous in the x variable by the proof of Lemma 3.4.1, we can approximate the integral of $\mathcal{N}$ in spatial variable by Riemann sums. The Riemann sums converge since $\mathcal{N}$ is continuous in x. We can bound all Riemann sums by supremum of $\mathcal{N}$ times size of the domain. Also observe that $\sup_{x\in[0,2\pi]} \mathcal{N}(t_0,t,x)$ has finite expectation; this follows again from use of Kolmogorov's continuity theorem (3.19). We can use conditional dominated convergence theorem [36, p. 121] to conclude that (3.43) hold. From (3.40),(3.42) and (3.43), we get

$$\mathrm{E}\left[V(t+t_0)|\mathscr{F}_{t_0}\right] \leq V(t_0)\ ,$$

which implies that $U(t)$ is a local supermartingale up to time $\mathfrak{t}$. □

A version of Doob-Meyer decomposition theorem in Appendix A tells us that V_N can be decomposed into nonincreasing continuous part $A_{N,t}$ and a continuous local martingale $M_{N,t}$. We have not yet justified that V_N is of class DL, which would be true if for example

$$\mathrm{E}\left[\sup_{t\leq a} V_N(t)\right] < \infty, \tag{3.44}$$

for every $a > 0$. From application of Kolmogorov's continuity theorem in section 3.4.3, we also get that $\mathrm{E}\left[\sup_{t\leq a, x\in[0,2\pi]} u_N(t,x)\right] \leq \infty$. One direct computation gives us

$$\mathrm{E}\left[\sup_{t\leq a} V_N(t)\right] = \mathrm{E}\left[\sup_{t\leq a}\int_0^{2\pi} u_N(t,x)\right] \leq \mathrm{E}\left[\sup_{t\leq a, x\in[0,2\pi]} u_N(t,x)\right] < \infty,$$

therefore V_N is of class DL. From uniqueness in Doob-Meyer decomposition, we have that if $\tilde{N} > N$, then $M_{\tilde{N},t} = M_{N,t}$ almost surely for $t \leq \mathfrak{t}_N$. We can define a local martingale M up to time $\mathfrak{t}$.

We know [51, V Prep. 1.8] that $\liminf_{t\to\mathfrak{t}} M_t = -\infty$ almost surely on a set $\{\langle M, M\rangle_{\mathfrak{t}} = \infty\}$. But U is positive; this implies that $\{\langle M, M\rangle_{\mathfrak{t}} = \infty\}$ has measure zero. On $\{\langle M, M\rangle_{\mathfrak{t}} < \infty\}$, the local martingale M_t converges almost surely [51, IV Prep. 1.26] as $t \to \mathfrak{t}$. Since M is continuous, we can conclude that $\sup_{t<\mathfrak{t}} U(t)$ is bounded by some almost surely finite random variable K.

3.6.1 Strong Markov Property

It is well known that the solution to the Stochastic Heat Equation has the Markov property. This is because

$$u_N(t+s,x) = (p_s * u_N(t,\cdot))(x) + \int_0^s\int_0^{2\pi} p_{s-l}(x-y)\sigma(u_N(t+l,y)\wedge N)\eta(dy,dl+t), \tag{3.45}$$

which means that we can start the solution anew at any time t. We will show that the process $u_N(t,\cdot)$ which takes values in a space $\mathcal{J}$ of positive continuous functions on a circle has the strong Markov property. This would be equivalent to replacing t in (3.45) with some stopping time τ.

The strong Markov property follows from the proof of [51, III Thm. 3.1], which we will detail in the next couple of lines. Suppose that some finite stopping time τ that takes only discrete values is some set D, then

$$\begin{aligned}
&u_N(\tau+t,x)\\
&=\sum_{t_d\in D}\mathbb{1}_{\tau=t_d}\left(\left(p_{t_d+t}*u(0,\cdot)\right)(x)+\int_0^{t+t_d}\int_0^{2\pi}p_{t_d+t-s}(x-y)\sigma(u_N(s,y)\wedge N)\eta(dy,ds)\right)\\
&=\sum_{t_d\in D}\mathbb{1}_{\tau=t_d}\left(\left(p_t*u(t_d,\cdot)\right)(x)+\int_0^{t}\int_0^{2\pi}p_{t-s}(x-y)\sigma(u_N(t_d+s,y)\wedge N)\eta(dy,ds+t_d)\right)\\
&=(p_t*u(\tau,\cdot))(x)+\int_0^{t}\int_0^{2\pi}p_{t-s}(x-y)\sigma(u_N(\tau+s,y)\wedge N)\eta(dy,ds+\tau) \qquad (3.46)
\end{aligned}$$

almost surely by the use of identity (3.45). As with the similar types of arguments, we will assume that τ is almost surely finite stopping and construct the following approximation which takes only discrete values: $\tau_k=\frac{\lfloor 2^k\tau\rfloor+1}{2^k}$. Now, we can take a limit as $k\uparrow\infty$ in (3.46). We immediately get that

$$\begin{gathered}
u_N(\tau_k,x)\to u_N(\tau,x)\\
(p_t*u(\tau_k,\cdot))(x)\to(p_t*u(\tau,\cdot))(x),
\end{gathered}$$

almost surely as $k\uparrow\infty$. The stochastic integral takes a little bit more work; a similar approach to the one in section 3.4.2 yields convergence to the correct quantity, that is

$$\int_0^t\int_0^{2\pi}p_{t-s}\sigma(u_N(\tau+s,y)\wedge N)\eta(dy,ds+\tau),$$

in $L^2(P)$ norm. Now take sub-sequence of k, denoted by k_j such that the stochastic integral converges almost surely [54, Riesz-Fisher, p. 148]. Get strong Markov property

$$u_N(\tau+s,x)=(p_s*u_N(\tau,\cdot))(x)+\int_0^s\int_0^{2\pi}p_{s-l}(x-y)\sigma(u_N(\tau+l,y)\wedge N)\eta(dy,dl+\tau), \tag{3.47}$$

which holds almost surely for almost surely finite stopping time τ.

3.7 Proof of Theorem 3.1.1

We would like to show that, if the solution u is large, then the convolution with p will bring it down and there is not much of a chance for the solution to grow larger. Define

$$M(t)=\sup_{x\in[0,2\pi]}u(t,x),$$

and

$$\tau_1 = \inf\{t > 0 : M(t) > 2^n\} \text{ where } 2^n > \sup_x g(x),$$
$$\tau_n = \inf\{t > \tau_{n-1} : M(t) > 2M(\tau_{n-1}) \text{ or } M(t) < M(\tau_{n-1})\}.$$

We picked the first stopping time to be the first hitting time of a level above the supremum of the initial condition. Since $p_t(x) \leq \text{const} \cdot t^{-1/\beta}$, because of (3.13), we have that

$$\int_0^{2\pi} p_t(x-y)u(t,y)dy \leq \text{const} \cdot t^{-1/\beta}U(t) \leq \text{const} \cdot t^{-1/\beta}K, \tag{3.48}$$

where the right-hand side of (3.48) is less than $M(\tau_n)/4$ if $t > \text{const} \cdot K^\beta M(\tau_n)^{-\beta}$. Choose and fix $K_0 > 0$ and define an event

$$F(n) = \left\{ \text{For } 0 \leq t \leq T(n),\ x \in [0, 2\pi] \text{ we have } \mathcal{N}(\tau_n, t, x) \leq \frac{1}{4}M(\tau_n) \right\}$$

where

$$T(n) = \text{const} \cdot K_0^\beta M(\tau_n)^{-\beta} \ .$$

If $M(\tau_n) = 2^l$, $N = 2^{l+1}$,$\Delta = 2^{l-2}$ and $T(n) = K_0^\beta 2^{-l\beta}$, then

$$\begin{aligned} \mathrm{P}\left(F(n) |\, M(\tau_n) = 2^l \right) &\geq 1 - \text{const} \left(\frac{N^\gamma T^{q(1-1/\beta-3\theta)/2}}{\Delta} \right)^k \\ &\geq 1 - \text{const} \left(2^{l(\gamma - q(\beta-1-3\theta\beta)/2-1)} \right)^k, \end{aligned} \tag{3.49}$$

for $M(\tau_n) = 2^l$ sufficiently large by Lemma 3.4.1. Which also means that, if $M(\tau_n)$ is high, then the probability that $M(\tau_{n+1}) = M(\tau_n)/2$ is also high. It is, from some point, greater than 1/2 provided that

$$\gamma < 1 + q(\beta - 1 - 3\theta\beta)/2 \ , \tag{3.50}$$

in the exponent of (3.49). Since θ can be taken arbitrarily close to zero and q can be taken arbitrarily close to one, then the maximal γ will be

$$\gamma < 1 + (\beta - 1)/2.$$

So far we have shown that the solution will not blow up a.s. on $\omega \in \{\omega : K(\omega) < K_0\}$, since the probabilities of growing larger are strictly smaller than 1/2. Now think of behavior of a random walk with drift at infinity. Since the random variable K is almost surely finite, it follows that

$$\mathrm{P}\left(\bigcup_{K_0\in\mathbb{N}}\{\omega : K(\omega) < K_0\}\right) = 1.$$

In other words, we can write out the probability space as a countable union of events $\{\omega : K(\omega) < K_0\}$. Note that the set on which the solution does not blow up and a set $\{\omega : K(\omega) < K_0\}$ only differ by a null set; we have a countable number of null sets which is still a null set.

We have justified that if the solution is high, then there is only a small chance for it to grow even higher. This does not [*yet*] imply that $\mathfrak{t} = \infty$. It might be possible for the solution to blow up in finite time if $\tau_{n+1} - \tau_n$ gets small with n. We will show that this cannot happen.

Suppose that $M(\tau_n) = 2^l$ for some $l \in \mathbb{Z}$, and let ς_n be the first time after τ_n when M reaches level 2^{l+1}. We have that

$$\int_0^{2\pi} p_t(x-y)u_N(\tau_n, y)dy \le M(\tau_n),$$

and because of strong Markov property and by Lemma 3.4.1,

$$\mathrm{P}\left(\sup_{x,0\le s\le t}|\mathcal{N}(\tau_n, s, x)| < 2^l\right) \ge \inf_{u_0\in\mathcal{J}}\mathrm{P}\left(\sup_{x,0\le s\le t}|\mathcal{N}(s,x)| < 2^l\right) \ge 1 - c\left(\frac{2^{\gamma(l+1)}t^{(1-1/\beta-\epsilon)/2}}{2^l}\right).$$

In particular,

$$\mathrm{P}\left(\varsigma_n - \tau_n > t\right) \ge 1 - c\left(\frac{2^{\gamma(l+1)}t^{(2-\epsilon-\alpha)/4}}{2^l}\right), \tag{3.51}$$

and thus $\varsigma_n - \tau_n$ is positive with probability one. The meaning of (3.51) is this: with each visit of level 2^l, it takes positive time to reach level 2^{l+1}. Also note that this positivity has structure (3.51) for each visit of level 2^l. The sum of infinitely many positive random variables with property (3.51) diverges with probability one. This also concludes the proof or Theorem 3.1.1. It must be that $\mathfrak{t} = \infty$ a.s., and the solution exists for all times.

Table 3.1. Table of some Lévy measures and corresponding Lévy exponents

Lévy measure	**Lévy exponent (proportional)**	**Constraint**
$\lvert z\rvert^{-1-\alpha}\,dz$	ξ^{α}	$\alpha \in (1,2)$
$e^{-\lvert z\rvert}\,\lvert z\rvert^{-1-\alpha}\,dz$	$-(1+\xi^2)^{\alpha/2}\cos(\alpha\arctan(\xi))$	
$\cos(z)\,\lvert z\rvert^{-1-\alpha}\,dz$	$(-\lvert -1+\xi\rvert^{\alpha}-\lvert 1+\xi\rvert^{\alpha})\cos(\pi\alpha/2)$	
$\sin(z)\,\lvert z\rvert^{-2-\alpha}\,dz$	$\lvert -1+\xi\rvert^{\alpha}+\xi\,\lvert -1+\xi\rvert^{\alpha}-\lvert 1+\xi\rvert^{\alpha}-\xi\,\lvert 1+\xi\rvert^{\alpha}$	

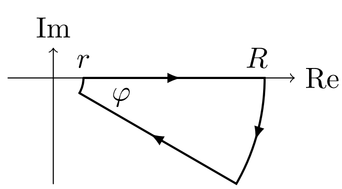

Figure 3.1. Contour for the asymptotic behavior of $q_t^{(j)}(x)$

CHAPTER 4

BLOW-UP

This chapter immediately follows Chapter 3, we will show blow-up in the case that $\mathcal{L}$ is the generator of a symmetric α-stable Lévy process with $\alpha \in (1,2)$. The condition on nonlinearity σ is taken to be

$$\sigma(x) = a\,|x|^{\gamma}. \tag{BL}$$

Here is the main theorem:

Theorem 4.0.1 (Blow-up)**.** *Suppose that $\mathcal{L}$ is a generator of a symmetric α stable Lévy process with $\alpha \in (1,2)$. Moreover, assume that σ has a form given by* (BL) *with $\gamma > \gamma_1 \vee \gamma_2$ where*

$$\begin{aligned} \gamma_1 &= 1 + \frac{\alpha-1}{2} = \frac{1+\alpha}{2} \\ \gamma_2 &= 1 + \frac{1}{2\alpha}, \end{aligned}$$

then the solution to (SHE) *blows-up in finite time with positive probability.*

Results of the above Theorem 4.0.1 and Theorem 3.1.1 are summarized in Figure 4.1. Quite interestingly, the coefficient α where curves $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$ intersect is equal to the golden ratio.

The proof of blow-up, which uses the techniques of Mueller and Sowers [43, 42] highly depends on the scaling properties of p_t. That is why we assumed $\mathcal{L}$ to be the generator of a symmetric α-stable process. First, we will outline the proof, then present necessary theorems and Lemmas in the section 4.1. Finally, in section 4.2, we will prove Theorem 4.0.1.

The proof goes as follows: we will show that the solution to (SHE) gets large with positive probability q. When that happens, we *'chop-up'* and rescale the solution into $\mathfrak{N}$ pieces which evolve almost independently. In the next step, each of $\mathfrak{N}$ subsolutions has a probability q to get large in some bounded time. This can be compared to a Galton-Watson (GW) branching process, since each subsolution has probability q to give a birth to another

$\mathfrak{N}$ subsolutions. It is known [62] that probability of survival of such a branching process is positive if

$$\mathfrak{N}q > 1. \tag{4.1}$$

The rescaling that we will introduce in the next few lines ensures that with each successive generation, the peaks grow larger. We will see that time also scales and an infinite number of generations of the GW process fits into bounded time interval.

4.1 Supporting Lemmas and theorems

The splitting part of our argument will require a slight modification of our equation to

$$\begin{aligned} \partial_t u &= \mathcal{L}u + b(u,\xi)\eta \\ u(0,x) &= g(0) \end{aligned} \tag{4.2}$$

where $b(x,y) = \sqrt{(x+y)^{2\gamma} - y^{2\gamma}}$ and ξ is a stochastic process, independent of sigma algebra generated by η and taking values in positive continuous functions on a circle. The technique of the proof follows general directions outlined in [43, 42]. Throughout this whole chapter, we will assume that

$$\mathfrak{t}(u) = \lim_{N\to\infty} \mathfrak{t}_N(u) = \infty \text{ a.s.,} \tag{4.3}$$

where

$$\mathfrak{t}_N(u) = \inf\{t > 0 : \sup_x u(t,x) \geq N\}, \tag{4.4}$$

and arrive at a contradiction. Let us think about the interpretation of (4.2) for a second. Similarly to the approach in Chapter 3, we can stop u whenever it reaches level N and ξ whenever it reaches level $\tilde{N}$. Solution to such an equation is surely positive by modification of the argument in section 3.5. Now send $\tilde{N}$ to infinity and get solution until the stopping time $\mathfrak{t}_N(u) \wedge \mathfrak{t}(\xi)$. Therefore, there exists a positive solution to (4.2) up to the stopping time $\mathfrak{t}(u) \wedge \mathfrak{t}(\xi)$. The solution is measurable with respect to the sigma algebra that contains sigma algebra generated by white noise η and stochastic process ξ.

Throughout this section, we will take $T = 1$ unless specified otherwise. Let us introduce the scaling Lemma.

Lemma 4.1.1 (Scaling)**.** *Let u be a solution to* (4.2) *on a circle of size Λ then*

$$v(t,x) := \mathbf{L}^{-1}u(\mathbf{L}^{-\alpha\lambda}t, \mathbf{L}^{-\lambda}x),$$

solves

$$\begin{aligned}\partial_t v &= \mathcal{L}v + b(v,\tilde{\xi})\tilde{\eta}\\ v(0,x) &= \mathbf{L}^{-1}u(0,x\mathbf{L}^{-\lambda}) = \mathbf{L}^{-1}g(x\mathbf{L}^{-\lambda})\end{aligned}$$

on a circle of size $\Lambda\mathbf{L}^{\lambda}$ for $\tilde{\xi}(t,x) = \mathbf{L}^{-1}\xi(\mathbf{L}^{-\alpha\lambda}t, \mathbf{L}^{-\lambda}x)$ and white noise

$$\tilde{\eta}(A) = \mathbf{L}^{(\alpha+1)\lambda/2}\int \mathbb{1}_A(s\mathbf{L}^{\alpha\lambda}, y\mathbf{L}^{\lambda})\eta(ds,dy).$$

The scaling parameter $\mathbf{L}$ is arbitrary, such that $\mathbf{L} \gg 1$ and λ is

$$\lambda = \frac{2\gamma-2}{\alpha-1}.$$

Proof. The scaling follows from writing down the mild solution and performing change of variables. The only thing we need to check is the scaling of $p_t(x;\Lambda)$ with respect to space and time:

$$\begin{aligned}p_{b^\alpha t}(x;\Lambda) &= \sum_{j\in\mathbb{Z}}\tilde{p}_{b^\alpha t}(x+j\Lambda) = \sum_{j\in\mathbb{Z}}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi(x+j\Lambda)-b^\alpha t|\xi|^\alpha}d\xi\\ &= \sum_{j\in\mathbb{Z}}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi(x+j\Lambda)-t|b\xi|^\alpha}d\xi = \frac{1}{b}\sum_{j\in\mathbb{Z}}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi(x/b+j\Lambda/b)-t|\xi|^\alpha}d\xi\\ &= \frac{1}{b}p_t(x/b;\Lambda/b).\end{aligned}$$

Write down the mild solution, using that $\tilde{t} = \mathbf{L}^{\alpha\lambda}t$, $\tilde{s} = \mathbf{L}^{\alpha\lambda}s$, $\tilde{x} = \mathbf{L}^{\lambda}x$, $\tilde{y} = \mathbf{L}^{\lambda}y$ and for simplicity, assume that $\xi \equiv 0$ to get

$$\begin{aligned}v(\tilde{t},\tilde{x}) &= \mathbf{L}^{-1}u(\mathbf{L}^{-\alpha\lambda}\tilde{t}, \mathbf{L}^{-\lambda}\tilde{x})\\ &= \mathbf{L}^{-1}\int_0^{\Lambda} p_t(x-y;\Lambda)u(0,y)dy + \mathbf{L}^{-1}\int_0^t\int_0^{\Lambda} p_{t-s}(x-y)u^{\gamma}(s,y)\eta(ds,dy).\end{aligned}$$

All we have left is to use the change of variables formula. The first integral will be

$$\begin{aligned}&\int_0^{\Lambda} p_{\mathbf{L}^{-\alpha\lambda}\tilde{t}}(x-y;\Lambda)\mathbf{L}^{-1}u(0,\mathbf{L}^{-\lambda}\mathbf{L}^{\lambda}y)dy = \int_0^{\Lambda} p_{\tilde{t}}(\mathbf{L}^{\lambda}x-\mathbf{L}^{\lambda}y)\mathbf{L}^{-1}u(0,\mathbf{L}^{-\lambda}\mathbf{L}^{\lambda}y)\mathbf{L}^{\lambda}dy\\ &= \int_0^{\tilde{\Lambda}} p_{\tilde{t}}(\tilde{x}-\tilde{y})\mathbf{L}^{-1}u(0,\mathbf{L}^{-\lambda}\tilde{y})d\tilde{y} = (p_{\tilde{t}} * v)(\tilde{x}).\end{aligned}$$

The stochastic integral will be

$$\mathbf{L}^{-1+\gamma}\int_0^{\mathbf{L}^{-\alpha\lambda}\tilde{t}}\int_0^{\mathbf{L}^{-\lambda}\tilde{\Lambda}} p_{\mathbf{L}^{-\alpha\lambda}\tilde{t}-\mathbf{L}^{-\alpha\lambda}\tilde{s}}(\mathbf{L}^{-\lambda}\tilde{x}-\mathbf{L}^{-\lambda}\tilde{y})\left(\mathbf{L}^{-1}u(\mathbf{L}^{-\alpha\lambda}\tilde{s},\mathbf{L}^{-\lambda}\tilde{y})\right)^{\gamma}\eta(ds,dy)$$
$$=\mathbf{L}^{-1+\gamma+\lambda}\int_0^{\mathbf{L}^{-\alpha\lambda}\tilde{t}}\int_0^{\mathbf{L}^{-\lambda}\tilde{\Lambda}} p_{\tilde{t}-\tilde{s}}(\tilde{x}-\tilde{y})v(\tilde{s},\tilde{y})^{\gamma}\eta(ds,dy)$$
$$=\mathbf{L}^{-1+\gamma+\lambda-(\alpha+1)\lambda/2}\int_0^{\tilde{t}}\int_0^{\tilde{\Lambda}} p_{\tilde{t}-\tilde{s}}(\tilde{x}-\tilde{y})v(\tilde{s},\tilde{y})^{\gamma}\tilde{\eta}(d\tilde{s},d\tilde{y}).$$

The condition on λ finishes the proof and we have

$$v(\tilde{t},\tilde{x})=(p_{\tilde{t}}*v)(\tilde{x})+\int_0^{\tilde{t}}\int_0^{\tilde{\Lambda}} p_{\tilde{t}-\tilde{s}}(\tilde{x}-\tilde{y})v(\tilde{s},\tilde{y})^{\gamma}\tilde{\eta}(d\tilde{s},d\tilde{y}).$$

□

Define the following local martingale

$$M(t)=\int_0^{\Lambda} p_{2T-s}(x;\Lambda)u(t,x)dx, \tag{4.5}$$

for $t\in[0,1]$. One can easily check, by writing down the mild solution, that $M(t)$ is indeed a local martingale up to $\mathfrak{t}(u)$. We will investigate formation of large peaks of u by looking at the martingale M. Large M implies large u. The probability that M gets large will play the role of q in (4.1).

4.1.1 Estimates and constants

We will require one more estimate for the integral of the density function p_t and a definition for a few constants. There exists a constant $C(a,T)$, for $a>1/(1+\alpha)$, fixed $T\geq 1$ and all $t\in[0,T]$ such that

$$\sup_{\Lambda>2\pi}\sup_{t\in[0,1]}\int_0^{\Lambda} p^a_{2T-t}(x;\Lambda)dx\leq C(a,T). \tag{4.6}$$

The requirement on a follows from integrability of $\tilde{p}$. Do not forget that $\tilde{p}_{2T-t}(x)=\mathcal{O}(|x|^{-(1+\alpha)})$; that is why we need to consider $a>1/(1+\alpha)$. This integrability issue for small a turns out to be the source for condition on γ_2 in Theorem 4.0.1. Constants that will be used are

$$\mathfrak{D}=\sup_{\Lambda>2\pi}\sup_{t\in[1,2]}\sup_{x\in\Lambda} p_t(x;\Lambda)=p_1(0;2\pi), \tag{4.7}$$
$$\mathfrak{E}:=\tilde{p}_1(1)\leq\inf_{t\in[1,2]}\inf_{\Lambda>2\pi}\inf_{x\in[-1,1]} p_t(x), \tag{4.8}$$
$$\mathfrak{G}=3/\mathfrak{E}, \tag{4.9}$$
$$\mathfrak{K}:=\mathfrak{G}(\mathfrak{D}+1), \tag{4.10}$$

and finally

$$\hat{q}_i = (q_{i+1} - q_i)/2.$$

Their meaning will be clear throughout the proof in section 4.1.2.

4.1.2 Probability of forming a peak

We will estimate the probability of M being large before time T by comparing it to a time-changed Brownian motion. Let us assume that $M(0) = 2A$, where A is some constant that we will specify at the end of this section. Define τ to be a stopping time such that

$$\tau = \inf\{t : 0 \le t \le T \wedge \mathfrak{t} \text{ and } M(t) \text{ is either } A \text{ or } \mathbf{L}A\mathfrak{K}\}, \tag{4.11}$$

where we take $\inf\{\emptyset\} = +\infty$. Stopping time τ will be the first time $M(t)$ reaches levels A or $\mathbf{L}A\mathfrak{K}$. We would like to show that $M(t)$ gets large with some positive probability before the time T.

Lemma 4.1.2. *If $\gamma > \gamma_2(\alpha)$, then*

$$\mathrm{P}(\tau > 1) \le \frac{1}{2\mathbf{L}\mathfrak{K}},$$

where

$$\gamma_2(\alpha) = 1 + \frac{1}{2\alpha}.$$

Proof. Throughout the proof, we will use a comparison to the Brownian motion and the reflection principle. The quadratic variation of M is

$$\langle M \rangle_t = \int_0^t \int_0^\Lambda p^2_{2T-s}(x) u^{2\gamma}(s,x) ds dx.$$

Use Jensen's inequality with respect to the probability measure $\frac{p^a_{2T-s}(x)}{\|p^a_{2T-s}\|_{L^1(\Lambda)}} dx$, where $L^p(\Lambda) \equiv L^p([0,\Lambda])$ to get

$$\begin{aligned}
\int_0^\Lambda p^2_{2T-s}(x) u^{2\gamma}(s,x) ds &= \left\|p^a_{2T-s}\right\|_{L^1(\Lambda)} \int_0^\Lambda p^{2-a}_{2T-s}(x) u^{2\gamma}(s,x) \frac{p^a_{2T-s}(x)}{\left\|p^a_{2T-s}\right\|_{L^1(\Lambda)}} dx \\
&\ge \left\|p^a_{2T-s}\right\|^{1-2\gamma}_{L^1(\Lambda)} \left(\int_0^\Lambda \left(p_{2T-s}(x)\right)^{(2-a+2a\gamma)/(2\gamma)} u(s,x) dx \right)^{2\gamma} \\
&\ge \left\|p^a_{2T-s}\right\|^{1-2\gamma}_{L^1(\Lambda)} \left(\int_0^\Lambda p_{2T-s}(x) u(s,x) dx \right)^{2\gamma} = \left\|p^a_{2T-s}\right\|^{1-2\gamma}_{L^1(\Lambda)} M(s)^{2\gamma},
\end{aligned}$$

provided that

$$(2 - a + 2a\gamma)/(2\gamma) = 1, \quad \text{or equivalently} \quad a = \frac{2\gamma - 2}{2\gamma - 1}.$$

The $L^1(\Lambda)$ norm of density p_t to power a is finite if $a > 1/\alpha$, which in turn gives us the following condition on γ:

$$\gamma > \gamma_2(\alpha) \quad \text{where} \quad \gamma_2(\alpha) = 1 + \frac{1}{2\alpha}. \tag{4.12}$$

We finally get an estimate on $\langle M\rangle_t$ as

$$\langle M\rangle_t \geq C(T,a)^{1-2\gamma} \int_0^t M(s)^{2\gamma} ds.$$

Up to time $\tau \wedge 1$, we have $M(t) > A$. This yields

$$\langle M\rangle_t \geq C_2 A^{2\gamma} t,$$

and if $\tau > 1$, then we get

$$\langle M\rangle_1 \geq C_2 A^{2\gamma}.$$

The continuity of M follows from the proof of Lemma 3.4.1. Every continuous local martingale $M(t)$ can be thought of as a time-changed Brownian motion [51, V Thm. 1.3]; that is $M(t) = B_{\langle M\rangle_t}$ for some Brownian motion B_t. Another direct computation gives us

$$\begin{aligned}
\mathrm{P}(1 < \tau) &= \mathrm{P}(1 < \tau, A < M(t) < \mathbf{L}A\mathfrak{K} \text{ for } t \in [0,1]) \\
&= \mathrm{P}(1 < \tau, A < 2A + B_{\langle M\rangle_t} < \mathbf{L}A\mathfrak{K} \text{ for } t \in [0,1]) \\
&= \mathrm{P}(1 < \tau, A < 2A + B_t < \mathbf{L}A\mathfrak{K} \text{ for } t \in [0,\langle M\rangle_1]) \\
&\leq \mathrm{P}(1 < \tau, A < 2A + B_t < \mathbf{L}A\mathfrak{K} \text{ for } t \in [0,C_2A^{2\gamma}]) \\
&\leq \mathrm{P}\left(\sup_{t\in[0,C_2A^{2\gamma}]} B(t) < (\mathbf{L}\mathfrak{K} - 2)A\right)
\end{aligned}$$

We can use reflection principle [51, Ch. III,Prep. 3.7] to further write

$$\begin{aligned}
\mathrm{P}(1 < \tau) &\leq 1 - \mathrm{P}\left(\sup_{t\in[0,C_2A^{2\gamma}]} B(t) > (\mathbf{L}\mathfrak{K} - 2)A\right) \\
&\leq 1 - 2 \cdot \mathrm{P}\left(B(C_2A^{2\gamma}) > (\mathbf{L}\mathfrak{K} - 2)A\right) \\
&\leq \mathrm{P}\left(|B(C_2A^{2\gamma})| < (\mathbf{L}\mathfrak{K} - 2)A\right) \\
&\leq \int_{-(\mathbf{L}\mathfrak{K}-2)A}^{(\mathbf{L}\mathfrak{K}-2)A} \frac{1}{\sqrt{2\pi C_2 A^{2\gamma}}} \exp\left(-\frac{x^2}{2C_2A^{2\gamma}}\right) dx \\
&\leq 2(\mathbf{L}\mathfrak{K} - 2)A\left(2\pi C_2 A^{2\gamma}\right)^{-1/2} \leq 2(2\pi C_2)^{-1/2}\mathbf{L}\mathfrak{K}A^{1-\gamma}.
\end{aligned}$$

The appropriate choice of

$$A = \left(2^2(2\pi C_2)^{-1/2}\mathbf{L}^2\mathfrak{K}^2\right)^{1/(\gamma-1)}, \tag{4.13}$$

concludes the proof of the Lemma. □

Lemma 4.1.3. *For $\gamma > \gamma_2(\alpha)$ and A defined in* (4.13)*, we have*

$$\mathrm{P}\left(M(\tau \wedge 1) \geq A\mathfrak{K}\mathbf{L}\right) \geq \frac{1}{2\mathfrak{K}\mathbf{L}},$$

where $\gamma_2(\alpha)$ is defined in (4.12).

Proof. Let us start the proof with the following observation. Since $M(0) = 2A$, the optional stopping theorem [51, II Thm. 3.2] implies that

$$2A = \mathrm{E}[M(0)] = \mathrm{E}[M(1 \wedge \tau \wedge \mathfrak{t}_N)].$$

Since $M(1 \wedge \tau \wedge \mathfrak{t}_N)$ is bounded, the bounded convergence convergence theorem implies that

$$2A = \mathrm{E}[M(0)] = \mathrm{E}[M(1 \wedge \tau)],$$

because we have assumed that $\mathfrak{t} = \infty$ almost surely. Another direct computation gives us

$$\begin{aligned} 2A &= \mathrm{E}[M(1 \wedge \tau)\mathbb{1}_{\tau \leq 1}] + \mathrm{E}[M(1 \wedge \tau)\mathbb{1}_{\tau > 1}] \\ &\leq \mathbf{L}A\mathfrak{K} \cdot \mathrm{P}(M(1 \wedge \tau) = \mathbf{L}A\mathfrak{K}, \tau \leq 1) + A \cdot \mathrm{P}(M(1 \wedge \tau) = A, \tau \leq 1) + \mathbf{L}A\mathfrak{K} \cdot \mathrm{P}(\tau > 1) \\ &\leq \mathbf{L}A\mathfrak{K} \cdot \mathrm{P}(M(1 \wedge \tau) = \mathbf{L}A, \tau \leq 1) + A + \mathbf{L}A\mathfrak{K} \cdot \mathrm{P}(\tau > 1). \end{aligned} \tag{4.14}$$

Lemma 4.1.2 gives us $\mathrm{P}(\tau > 1) \leq \frac{1}{2\mathbf{L}\mathfrak{K}}$ and that concludes the proof of Lemma 4.1.3. □

4.1.3 Splitting Lemma

Once M reaches level $\mathbf{L}^{\lambda}\mathfrak{K}A$, we would like to have a way to rescale and split the solution into $\mathfrak{N}$ almost independent subsolutions, such that the martingales (4.5) for each subsolution start at level $2A$. Suppose that indeed, $M(t) \geq \mathbf{L}\mathfrak{K}A$. At time $t < 1$, we can start solution anew with an initial condition $g(x) = u(t, x)$. We have that

$$\int_0^{\Lambda} p_{2T-t}(x; \Lambda) g(x) dx \geq \mathbf{L}\mathfrak{K}A,$$
$$\int_0^{\Lambda} p_{2T-t}(x; \Lambda) \frac{1}{\mathbf{L}} g(x) dx \geq \mathfrak{K}A,$$
$$\int_0^{\tilde{\Lambda}} p_{2T-t}(x\mathbf{L}^{-\lambda}; \Lambda) \frac{1}{\mathbf{L}} g(x\mathbf{L}^{-\lambda}) dx \geq \mathbf{L}^{\lambda}\mathfrak{K}A.$$

We can continue the solution by Lemma 4.1.1, where $g(x\mathbf{L}^{-\lambda})/\mathbf{L}$ will be the new initial condition. Instead, we will split the initial condition into $\mathfrak{N}$ pieces and look at how to form subsolutions. We can split $\tilde{g}(x) = g(x\mathbf{L}^{\lambda})/\mathbf{L}$ into $\tilde{g}(x) = \sum_{i=1}^{\mathfrak{N}} \tilde{g}_i(x)$ such that

$$\int_0^{\tilde{\Lambda}} p_{2T}(x; \tilde{\Lambda}) \tilde{g}_i(x) \geq 2A,$$

for every $i \in \{1, \ldots, \mathfrak{N}\}$. Functions $\tilde{g}_i$ will play the role of a new set of initial functions.

Lemma 4.1.4 (Splitting Lemma). *Let f_0 be a positive continuous function on a circle of size $\tilde{\Lambda} = \Lambda \mathbf{L}^{\lambda}$ such that*

$$\int_0^{\tilde{\Lambda}} p_{2T-t}(y\mathbf{L}^{-\lambda};\Lambda) f_0(y) dy \geq A\mathfrak{K}\mathbf{L}^{\lambda}, \tag{4.15}$$

$$\tag{4.16}$$

and $t \in [0,1]$, then there exists positive continuous functions $f_i, i \in \{1,\ldots,\mathfrak{N}\}$ such that

$$f_0(y) = \sum_{i=1}^{\mathfrak{N}} f_i(y),$$

and

$$\int_0^{\tilde{\Lambda}} p_{2T}(x - x_i;\tilde{\Lambda}) f_i(x) \geq 2A \text{ for } i \in \{1,\ldots,\mathfrak{N}\}.$$

Moreover, we have that $\mathfrak{N} = \lfloor \mathbf{L}^{\lambda} \rfloor - 1$ where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. Now, put points q_i on a circle such that $q_1 = 1$ and

$$q_{i+1} = \begin{cases} \min\left(\inf\{z > q_i : \int_{q_i}^{z} f_0(x)dx \geq A\mathfrak{G}\}, 1\right) & \text{if } \int_{q_i}^{\tilde{\Lambda}-1} f_0(x)dx \geq A\mathfrak{G} \\ \tilde{\Lambda} - 1 \text{ otherwise.} \end{cases}$$

Let I be the following set:

$$I = \left\{ i : \int_{q_i}^{q_{i+1}} f_0(x)dx \geq A\mathfrak{G} \right\}.$$

For every $i \in I$, we have

$$2A < 3A = \mathfrak{E}\mathfrak{G}A \leq \mathfrak{E}\int_{q_i}^{q_{i+1}} f_0(x)dx \leq \int_{q_i}^{q_{i+1}} p_{2T}(x - \hat{q}_i) f_0(x)dx. \tag{4.17}$$

We will show that I has *[almost]* the desired cardinality. Rewrite (4.15) as

$$\begin{aligned} A\mathfrak{K}\mathbf{L}^{\lambda} &\leq \int_{-1}^{1} p_{2T-t}(x\mathbf{L}^{-\lambda}) f_0(x)dx + \sum_{i\in I}\int_{q_i}^{q_{i+1}} p_{2T-t}(x\mathbf{L}^{-\lambda}) f_0(x)dx \\ &\quad + \sum_{i\notin I}\int_{q_i}^{q_{i+1}} p_{2T-t}(x\mathbf{L}^{-\lambda}) f_0(x)dx \\ A\mathfrak{K}\mathbf{L}^{\lambda} &\leq \mathfrak{D}\int_{-1}^{1} f_0(x)dx + \mathfrak{D}\mathfrak{G}A\,|I| + A\mathfrak{G}\int_0^{\tilde{\Lambda}} p_{2T-t}(x\mathbf{L}^{-\lambda})dx \\ A\mathfrak{K}\mathbf{L}^{\lambda} &\leq \mathfrak{D}\int_{-1}^{1} f_0(x)dx + \mathfrak{D}\mathfrak{G}A\,|I| + A\mathfrak{G}\mathbf{L}^{\lambda}. \end{aligned}$$

Using the definition of $\mathfrak{K}$, we get

$$\left(\mathfrak{K}/(\mathfrak{D}\mathfrak{G}) - 1/\mathfrak{D}\right)\mathbf{L}^{\lambda} \leq |I| + \frac{1}{\mathfrak{G}A}\int_{-1}^{1} f_0(x)dx,$$
$$\mathbf{L}^{\lambda} \leq |I| + \left\lceil \frac{1}{\mathfrak{G}A}\int_{-1}^{1} f_0(x)dx \right\rceil,$$

and in a case that $\int_{-1}^{1} f_0(x)dx = \beta\mathfrak{G}A$, the previous term will be

$$\mathbf{L}^{\lambda} \leq |I| + \lfloor\beta\rfloor + 1,$$
$$\mathbf{L}^{\lambda} - 1 \leq |I| + \lfloor\beta\rfloor.$$

The new variable β is defined as

$$\beta := \frac{1}{\mathfrak{G}A}\int_{-1}^{1} f_0(x)dx.$$

Now it is clear how we are going to create functions f_i. From properties of I, we have that for $i \in I$

$$\int_0^{\tilde{\Lambda}} p_{2T}(x - \hat{q}_i)\mathbb{1}_{(q_i,q_{i+1})}(x)f_0(x)dx > 3A.$$

Define function $f_i(x) := g_i(x)f_0(x)$, where g_i is a nonnegative continuous function bounded by $\mathbb{1}_{(q_i,q_{i+1})}(x)$ such that:

$$\int_0^{\tilde{\Lambda}} p_{2T}(x - \hat{q}_i)g_i(x)f_0(x)dx = \int_0^{\tilde{\Lambda}} p_{2T}(x - \hat{p}_i)f_i(x)dx > 2A.$$

Thus, the first $|I|$ functions f_i will be of form $g_i(x)f_0(x)$. The next $\lfloor\beta\rfloor$ of them can be created in a similar way. We have that

$$\int_0^{\tilde{\Lambda}} p_{2T}(x)f_0(x)dx = 3\beta A \geq 3\lfloor\beta\rfloor A,$$

and surely we can find continuous nonnegative functions $g_j, \operatorname{supp}(g_j) \subset [-1, 1]$ bounded by 1 with mutually disjoint supports such that for every $j \in \{1, \ldots, \lfloor\beta\rfloor\}$

$$\int_0^{\tilde{\Lambda}} p_{2T}(x)g_j(x)f_0(x)dx = \int_0^{\tilde{\Lambda}} p_{2T}(x)f_j(x)dx > 2A.$$

At this moment, there is a leftover positive piece

$$f_0(x) - \sum_{i\in I} f_i(x) - \sum_{j=1}^{\lfloor\beta\rfloor} f_j(x),$$

which can be added to any of the existing $f_i(x)$. We found $\mathfrak{N}$ $(= |I| + \lfloor\beta\rfloor)$ functions, where $\mathfrak{N} > \mathbf{L}^{\lambda} - 1$, and locations x_i with property

$$\int_0^{\tilde{\Lambda}} p_{2T}(x - x_i)f_i(x)dx > 2A,$$

where $x_i = \hat{q}_i$ for first $|I|$ of them. For the next $\lfloor\beta\rfloor$ of them, we will define $x_i = 0$. □

We have seen how to split the initial condition; now we will show how to produce $\mathfrak{N}$ almost independent solutions. Recall [43, Lem. 2.5].

Lemma 4.1.5 (C. Mueller and R. Sowers [43])**.** *Consider $\mathfrak{N}$ recursively defined equations*

$$\partial_t u_i(t,x) = \mathcal{L}u_i(t,x) + b\left(u_i, \sum_{j=1}^{i-1} u_j\right)\eta_i$$
$$u_i(0,x) = f_i(x) \tag{4.18}$$

where $u_0 \equiv 0$ on a circle of size Λ. Here the η_i are independent white noises and the $u_i(0,x)$ are nonnegative initial conditions. Let us define

$$\tilde{u}(t,x) := \begin{cases} \sum_{i=1}^{\mathfrak{N}} u_i(t,x) \text{ for } 0 \le t < \min\{\mathfrak{t}(u_i) : i = 1, \dots, \mathfrak{N}\} \\ \infty \text{ otherwise} \end{cases}$$

for all $t \ge 0$. The $\tilde{u}$ is a solution of

$$\partial_t \tilde{u} = \mathcal{L}\tilde{u} + \tilde{u}^\gamma \tilde{\eta},$$
$$\tilde{u}(0,x) = \sum_{i=1}^{\mathfrak{N}} u_i(0,x), \tag{4.19}$$

on a circle of size Λ for some white noise $\tilde{\eta}$.

Proof. Even though Lemma 4.1.4 is not identical to [43, Lem. 2.5], the proof works the same way. From the recursive nature of the theorem, it is enough to prove it for $\mathfrak{N} = 2$. The white noise will be defined as

$$\begin{aligned}\tilde{\eta}(A) := &\int_A \mathbb{1}\left(0 < u_1(s,y) + u_2(s,y), s < \mathfrak{t}(u_1) \wedge \mathfrak{t}(u_2)\right) \cdot \\ &\qquad \cdot \sqrt{\frac{(u_1(s,y) + u_2(s,y))^{2\gamma} - u_2(s,y)^{2\gamma}}{(u_1(s,y)+u_2(s,y))^{2\gamma}}}\eta_1(ds,dy) \\ &+ \int_A \mathbb{1}\left(0 < u_1(s,y) + u_2(s,y), s < \mathfrak{t}(u_1) \wedge \mathfrak{t}(u_2)\right)\left(\frac{u_1(s,y)}{u_1(s,y)+u_2(s,y)}\right)^\gamma \eta_2(ds,dy) \\ &+ \int_A \mathbb{1}\left(u_1(s,y) + u_2(s,y) = 0 \text{ or } t > \mathfrak{t}(u_1) \wedge \mathfrak{t}(u_2)\right)\eta_1(ds,dy).\end{aligned}$$

All we need to show is that if u_1 and u_2 solve (4.18), then $\tilde{u}$ solves (4.19). Write the stochastic integral for the mild solution of (4.19) and get

$$\begin{aligned}&\int_0^t \int_0^{2\pi} p_{t-s}(x-y)(u_1(s,y)+u_2(s,y))^\gamma \tilde{\eta}(ds,dy) \\ &= \int_0^t \int_0^{2\pi} p_{t-s}(x-y)\sqrt{(u_1(s,y)+u_2(s,y))^{2\gamma} - u_2(s,y)^{2\gamma}}\eta_1(ds,dy) \\ &\qquad + \int_0^t \int_0^{2\pi} p_{t-s}(x-y)\left(u_2(s,y)\right)^\gamma \eta_2(ds,dy) \\ &= u_2(t,x) + u_1(t,x) - (p_t * (u_1(0,\cdot) + u_2(0,\cdot))(x) = \tilde{u}(t,x) - (p_t * \tilde{u}(0,\cdot))(x)\end{aligned}$$

which concludes the theorem. $\square$

The previous theorem tells us that $\tilde{u}$, as defined in (4.19), solves the same equation as the equation of interest, but with different noise. We will need to establish that $\tilde{u}$ 'behaves' the same way as u.

Theorem 4.1.6 (Weak Uniqueness)**.** *Let u be a solution to* (SHE) *with $\mathcal{L}$ being the generator of an α-stable Lévy process with $\alpha \in (0,1)$ and $\sigma(x) = a\,|x|^{\gamma}$. Similarly, let $\tilde{u}$ be the solution to* (4.19)*, that is with the same generator but different noise $\tilde{\eta}$. Both u and $\tilde{u}$ share the same deterministic initial condition φ. Also, let u_N be defined as in* (CAPN) *and similarly for $\tilde{u}_N$. Then u_N has the same law as $\tilde{u}_N$ for all $N \in \mathbb{N}$.*

Proof. Take arbitrary $N \in \mathbb{N}$. All we need is that u_N and $\tilde{u}_N$ have the same finite dimensional distributions, that is

$$\mathrm{P}\left(u_N(t_1,x_1) \in A_1, \dots, u_N(t_k,x_k) \in A_k\right) = \mathrm{P}\left(\tilde{u}_N(t_1,x_1) \in A_1, \dots, \tilde{u}_N(t_k,x_k) \in A_k\right)$$

for every $t_j \in [0,\infty), x_j \in [0,2\pi], k \in \mathbb{N}$ and $A_j \in \mathscr{B}(\mathbb{R})$. One can easily see this from the construction of u_N and $\tilde{u}_N$ via Picard iterates. Kolmogorov's extension theorem in Appendix B tells us that both u_n and $\tilde{u}_N$ have the same law. We think of application of the theorem, as stated in Appendix B, with $E = [0,\infty)$ endowed with standard topology and $T = [0,\infty) \times [0,2\pi]$. $\square$

The previous theorem has additional implications. We have, for example

$$\mathrm{P}\left(\tau_N(u) < x\right) = \mathrm{P}\left(\tau_N(\tilde{u}) < x\right), \forall x \geq 0.$$

Or in other words, stopping times we are interested in have the same law even though both u and $\tilde{u}$ do not need to be defined on the same probability space.

At the moment, we have all the key estimates for the proof. The idea behind the proof is the following: Wait until the solution gets large, which happens with some positive probability according to Lemma 4.1.3. Once that happens, split and rescale the solution according to Lemmas 4.1.1 and 4.1.4. Those rescaled pieces will play the role of initial conditions in Lemma 4.1.5. Each subsolution u_i from Lemma 4.1.5 will evolve almost independently. We can again use Lemma 4.1.3 to get a positive probability of subsolution u_i forming a peak and continue the similar argument by again invoking Lemmas 4.1.1,4.1.4 and 4.1.5 again. This argument, of course, needs a rigorous treatment. The next section gives rigorous treatment to our argument.

4.2 Description of the branching process

First of all, we will need a way to encode generations of all subsolutions, births and deaths of branching process. As we stated earlier, solution of the original equation will give a rise to $\mathfrak{N}$ subsolutions. And each of those will possibly create $\mathfrak{N}$ new subsolutions in the subsequent generation. We will denote a particular solution in a particular generation by multi-index $\mathfrak{i} \in \mathfrak{Z}_n$, where $\mathfrak{Z}_n := \{1, \dots, \mathfrak{N}\}^n$ is n-fold Cartesian product. The set of all possible indices will be $\mathfrak{Z} := \cup_{n=1}^{\infty} \mathfrak{Z}_n$. Naturally, we have partial order on $\mathfrak{Z}$. Let $\mathfrak{i} = (i_1, \dots, i_n) \in \mathfrak{Z}_n$ and $j \in \mathfrak{Z}_m$; we will say that $\mathfrak{j}$ is a children of $\mathfrak{i}$, or $\mathfrak{i}$ is a parent of $\mathfrak{j}$ and denote

$$\mathfrak{i} \preceq \mathfrak{j}, \tag{4.20}$$

if $m \geq n$ and $\mathfrak{j} = (i_1, \dots, i_n, \dots)$.

We will also record whether subsolution $\mathfrak{i}$ got large by setting $\mathfrak{X}_\mathfrak{i}$ to one. If the subsolution indexed by $\mathfrak{i}$ did not get large, then we will set $\mathfrak{X}_\mathfrak{i}$ to zero. We will also set the subsequent generation to zero, that is we will take $\mathfrak{X}_\mathfrak{j} = 0$ for $\mathfrak{j} \succeq \mathfrak{i}$.

4.2.1 First generation

Recall the definition of the stopping time τ from (4.11). Let

$$\tilde{\tau} := \begin{cases} \tau \text{ if } M(\tau) = \mathbf{L}A\mathfrak{K} \\ \infty \text{ otherwise} \end{cases} \tag{4.21}$$

be the first time martingale M reaches level $\mathbf{L}A\mathfrak{K}$. If $\tilde{\tau} > 1$, then we can set $\mathfrak{X}_\mathfrak{i}$ to zero for every $\mathfrak{i} \in \mathfrak{Z}$. Now, suppose that $\tilde{\tau} \leq 1$, which happens with positive probability because of Lemma 4.1.3, then we can use $f_0(x) = u(\tilde{\tau}, x)$ as a new initial condition and restart the solution at that point. Use Lemma 4.1.4 to find a new set of initial conditions $f_i(x), i \in \{1, \dots, \mathfrak{N}\}$ on a circle of length $2\pi\mathbf{L}^\lambda$. Let us emphasize that those new initial conditions will depend on $u(\tilde{\tau}, \cdot)$. Finally, let $u_\mathfrak{i}, \mathfrak{i} \in \mathfrak{Z}_1$ be subsolutions on a circle of length $2\pi\mathbf{L}^\lambda$ as defined in Lemma 4.1.5. Then $\tilde{u} = \sum_{\mathfrak{i} \in \mathfrak{Z}_1} u_\mathfrak{i}$ is a solution to (4.19) on circle of length $2\pi\mathbf{L}^\lambda$ with respect to some white noise $\tilde{\eta}$. Solution $\tilde{u}$ is only a rescaled version of a solution to our original problem on a circle of length 2π by Lemma 4.1.1. Solution $\tilde{u}$ scaled back to circle of length 2π is not quite the same as u, but both have the same law by Theorem 4.1.6. For each $u_\mathfrak{i}$, we can define $\tau_\mathfrak{i}, \tilde{\tau}_\mathfrak{i}$ and $M_\mathfrak{i}$ similarly to (4.11), (4.21) and (4.5). If $M_\mathfrak{i}$ reaches level $\mathbf{L}A\mathfrak{K}$ before time one, then we set $\mathfrak{X}_\mathfrak{i}$ to one and continue the process which will be described in the next section. If $M_\mathfrak{i}$, for $\mathfrak{i} = (i_1)$, fails to reach $\mathbf{L}A\mathfrak{K}$, then we will set $\mathfrak{X}_{(i_1)}$ to zero. We will also set $\mathfrak{X}_\mathfrak{j}$ to zero for $\mathfrak{j} \succeq \mathfrak{i}$. Let us emphasize that if for any $\mathfrak{i} \in \mathfrak{Z}$, the martingale

$M_{\mathfrak{i}}$ fails to reach $\mathbf{L}A\mathfrak{K}$ before time one, we will not investigate subsequent generations and set $\mathfrak{X}_{\mathfrak{j}} = 0, \mathfrak{j} \succeq \mathfrak{i}$. We can define the *'solution after the first generation/splitting'* piecewise as

$$u_1(t,x) = \begin{cases} u(t,x) & \text{if } \tilde{\tau} \geq 1 \\ u(t,x) & \text{if } \tilde{\tau} < 1 \text{ and } t < \tilde{\tau} \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1} \mathbf{L}u_{\mathfrak{i}}(\mathbf{L}^{\alpha\lambda}(t-\tilde{\tau}), \mathbf{L}^{\lambda}x) & \text{if } \tilde{\tau} < 1 \text{ and } t \geq \tilde{\tau}, \end{cases}$$

where $u_{\mathfrak{i}}$ are obtained as described above. We have $\sum_{\mathfrak{i}\in\mathfrak{Z}_1} \mathbf{L}u_{\mathfrak{i}}(0, \mathbf{L}^{\lambda}x) = \mathbf{L}\tilde{u}(0, \mathbf{L}^{\lambda}x) = u(\tau, x)$. One might be worried what is going to happen if $u_{(1)}$ in u_1 get very large. The problem is that subsequent $u_{(2)}, u_{(3)}, \ldots, u_{(\mathfrak{N})}$ are defined using $u_{(1)}$. Explosion of $u_{(1)}$ cannot happen, since u_1 is the scaled sum of $u_{(i)}, i \in \{1, \ldots, \mathfrak{N}\}$. This would imply that $\mathrm{P}(\mathfrak{t}(u_1) < \infty) > 0$ and give us a contradiction with the assumption $\mathrm{P}(\mathfrak{t} = \infty) = 1$. By a similar argument, we can also assume that $\mathfrak{t}(u_{\mathfrak{i}}) = \infty$ almost surely for $\mathfrak{i} \in \mathfrak{Z}$.

We will further assume that the probability space Ω supports countable number of independent white noises $\eta_{\mathfrak{i}}, \mathfrak{i} \in \mathfrak{Z}$. Those noises come up naturally when we use Lemma 4.1.5.

4.2.2 Subsequent generations

At this point, we will need to introduce additional notation. Let $\mathbf{l}_{\mathfrak{i}}$ be the size of the circle for a particular generation $\mathfrak{i}$. We have that $\mathbf{l}_{\mathfrak{i}} = \mathbf{L}^{\lambda k}$ for $\mathfrak{i} \in \mathfrak{Z}_k$. We will need scaling for the time variable as well; let $\mathbf{s}_{\mathfrak{i}} = \mathbf{L}^{\alpha\lambda k}$ for $\mathfrak{i} \in \mathfrak{Z}_k$. Moreover, scaling for the height of the subsolution is also necessary. We will also define $\mathbf{a}_{\mathfrak{i}} = \mathbf{L}^k$ for $\mathfrak{i} \in \mathfrak{Z}_k$. If $\tilde{\tau} < 1$, then we get $\mathfrak{N}$ subsolutions $u_{\mathfrak{i}}, \mathfrak{i} \in \mathfrak{Z}_1$ and corresponding stopping times $\tilde{\tau}_{\mathfrak{i}}, \mathfrak{i} \in \mathfrak{Z}_1$. Martingales $M_{(j)}, j \in \{1, \ldots, \mathfrak{N}\}$ are not independent, but the corresponding Brownian motions (time changed M) are independent by Knight's theorem in Appendix B. Knight's theorem requires that $\langle M_{(i)}, M_{(j)} \rangle_t = 0$ for $i \neq j$, which is satisfied trivially since solutions $u_{(i)}$ and $u_{(j)}$ are driven by independent white noises. Suppose that subsolution indexed by 2 develops peak first, that is $\tilde{\tau}_{(2)} < 1$ and $\tilde{\tau}_{(2)} < \tilde{\tau}_{\mathfrak{i}}$ for $\mathfrak{i} \neq (2)$. This is a source of little problems since other $u_{\mathfrak{i}}, \mathfrak{i} = (3), (4), \ldots, (\mathfrak{N})$ are defined using $u_{(2)}$. Also, $u_{(1)}$ enters definition of $u_{(2)}$. We do not have a uniqueness result for the system of type (4.2). At point $\tilde{\tau}_{(2)}$, we should start solutions anew with now splitted $u_{(2)}$. We will need to keep track at which point that happened on the original timescale; thus define $v_{\mathfrak{i}} = \tilde{\tau} + \sum_{\mathfrak{j}\leq\mathfrak{i},\mathfrak{i}\neq\mathfrak{j}} \tilde{\tau}_{\mathfrak{j}}/\mathbf{s}_{\mathfrak{j}}$. With solution u_2 (the solution for second generation), we would like to fully describe the second generation of the branching process. That is, we would like to talk about probabilities of $u_{(i,j)}, (i,j) \in \mathfrak{Z}_2$ getting large. Now suppose that all $\tilde{\tau}_{\mathfrak{i}} < 1$ for $\mathfrak{i} \in \mathfrak{Z}_1$. There is $\mathfrak{N}!$ possible ways to order them. In addition, not all $u_{\mathfrak{i}}, \mathfrak{i} \in \mathfrak{Z}_1$ will survive. An easy combinatorial argument shows that there is

$$\binom{\mathfrak{N}}{0}\mathfrak{N}! + \binom{\mathfrak{N}}{1}(\mathfrak{N}-1)! + \cdots + \binom{\mathfrak{N}}{\mathfrak{N}-1}1! + \binom{\mathfrak{N}}{\mathfrak{N}}0! = \sum_{j=0}^{\mathfrak{N}} \binom{\mathfrak{N}}{j}(\mathfrak{N}-j)!$$

possible combinations of survival and arrival times. This added combinatorial difficulty causes that we will not be able to explicitly write the solution for the whole 'second generation' u_2. We can write the first couple of cases of u_2 as

$$u_2(t,x) = \begin{cases} u(t,x) & \text{if } t < \tilde{\tau} \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) & \text{if } \tilde{\tau} < 1 \text{ and } \tilde{\tau} \le t \le \min(\upsilon_{(1)}, \ldots, \upsilon_{(\mathfrak{N})}) \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1, \mathfrak{i}\neq(1)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) & + \sum_{\mathfrak{i}=(1,\cdot)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) \\ & \text{if } \tilde{\tau} < 1, \tilde{\tau}_{(1)} < 1, \upsilon_{(1)} = \min(\upsilon_{(1)}, \ldots, \upsilon_{(\mathfrak{N})}), \\ & \text{and } \upsilon_{(1)} \le t \le \min \upsilon_{(2)}, \ldots, \upsilon_{(n)} \\ \\ \quad \vdots \end{cases}$$

If $\tilde{\tau}_{(1)} = \min(\tilde{\tau}_{(1)}, \ldots, \tilde{\tau}_{(\mathfrak{N})}) < 1$, then solutions $u_{(1,j)}$ will be defined on a circle of size $2\pi\mathbf{L}^{2\lambda}$ and $u_{(1)}(\tilde{\tau}_{(1)}, x) = \sum_{j=1}^{\mathfrak{N}} \mathbf{L} u_{(1,j)}(0, \mathbf{L}^\lambda x)$ and also set $\mathfrak{X}_{(1)} = 1$.

The solution describing the third generation, called u_3, will need to consider all possible cases of births and deaths up to the second generation. It will also take into account the order of $\upsilon_\mathfrak{i}, \mathfrak{i} \in \bigcup_{j=1,2} \mathfrak{Z}_j$. To demonstrate this, assume that $\upsilon_{(1)} < \min(\upsilon_{(1)}, \ldots, \upsilon_{(\mathfrak{N})})$ and $\tilde{\tau}_{(1)} < 1$. In that case, at time $\upsilon_{(1)}$, we would split $u_{(1)}$ into subsolutions. From this point on, we will need to track solutions which are still *'alive'*, that is $u_{(2)}, \ldots, u_{(\mathfrak{N})}$ and $u_{(1,1)}, \ldots, u_{(1,\mathfrak{N})}$. The next splitting is going to occur at random time $\min(\upsilon_{(2)}, \ldots, \upsilon_{(\mathfrak{N})}, \upsilon_{(1,1)}, \ldots, \upsilon_{(1,\mathfrak{N})})$ assuming that $\min(\tilde{\tau}_{(2)}, \ldots, \tilde{\tau}_{(\mathfrak{N})}, \tilde{\tau}_{(1,1)}, \ldots, \tilde{\tau}_{(1,\mathfrak{N})}) < 1$. Let us write down the solution in the sub-branch, where $u_{(2)}$ gets large before time one in its own timescale first, that is $\upsilon_{(2)} = \min(\upsilon_{(2)}, \ldots, \upsilon_{(\mathfrak{N})}, \upsilon_{(1,1)}, \ldots, \upsilon_{(1,\mathfrak{N})})$. We can write just the first few above-described cases of 'third generation' solution u_3. That is:

$$u_3(t,x) = \begin{cases} u(t,x) & \text{if } t < \tilde{\tau} \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) & \text{if } \tilde{\tau} < 1 \text{ and } \tilde{\tau} \le t \le \min(\upsilon_{(1)}, \dots, \upsilon_{(\mathfrak{N})}) \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1, \mathfrak{i}\neq(1)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) & + \sum_{\mathfrak{i}=(1,\cdot)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) \\ & \text{if } \tilde{\tau} < 1, \tilde{\tau}_{(1)} < 1, \upsilon_{(1)} = \min(\upsilon_{(1)}, \dots, \upsilon_{(\mathfrak{N})}), \\ & \upsilon_{(1)} \le t, \\ & t < \min(\upsilon_{(2)}, \dots, \upsilon_{(\mathfrak{N})}, \upsilon_{(1,1)}, \dots, \upsilon_{(1,\mathfrak{N})}) \\ \sum_{\mathfrak{i}\in\mathfrak{Z}_1, \mathfrak{i}\neq(1)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) & + \sum_{\mathfrak{i}=(1,\cdot)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) \\ & + \sum_{\mathfrak{i}=(2,\cdot)} \mathbf{a}_\mathfrak{i} u_\mathfrak{i}(\mathbf{s}_\mathfrak{i}(t-\upsilon_\mathfrak{i}), \mathbf{l}_\mathfrak{i} x) \\ & \text{if } \tilde{\tau} < 1, \tilde{\tau}_{(1)} < 1, \upsilon_{(1)} = \min(\upsilon_{(1)}, \dots, \upsilon_{(\mathfrak{N})}), \\ & \tilde{\tau}_{(2)} < 1, \\ & \upsilon_2 = \min(\upsilon_{(2)}, \dots, \upsilon_{(\mathfrak{N})}, \upsilon_{(1,1)}, \dots, \upsilon_{(1,\mathfrak{N})}) \\ & t \ge \upsilon_{(2)} \\ \quad\vdots & \end{cases}$$

In the above-described scenario, we will record $\mathfrak{X}_{(2)} = 1$ if $\tilde{\tau}_{(2)} < 1$ and $\upsilon_{(2)} = \min(\upsilon_{(2)}, \dots,$ $\upsilon_{(\mathfrak{N})}, \upsilon_{(1,1)}, \dots, \upsilon_{(1,\mathfrak{N})})$. We can continue with the construction of u_4, u_5 and so on.

One can notice slight abuse of notation for stopping times $\tilde{\tau}_\mathfrak{i}$. We should also indicate which *'generation'* we have in mind. More precisely, we should write for example $\tilde{\tau}_{1,(1)}$ if we consider solution u_1; similarly, we should talk about $\tilde{\tau}_{2,(1)}$ if we consider u_2 and so on. We will omit the notation which would add corresponding *'splitting number'* to the definition of stopping time $\tilde{\tau}$. It is important to notice that splitting times $\tilde{\tau}$ will have the same distributions regardless of which generation we are talking about. That is all due to the Knight's theorem.

We can consider another construction of solutions u, u_1, etc. We can instead split subsolutions in every sub-branch it gets large. However, there is no guarantee that we will have a finite number of those splits in the original time scale; this is discussed in details in [43, p. 25]. We need to use the same approach as in [43], that is, we need to consider only a finite number of generations of the branching process.

Up to generation i, the probability of $\mathfrak{X}_\mathfrak{i}, \mathfrak{i} \in \bigcup_{j=1}^{i-1} \mathfrak{Z}_j$ giving birth to $\mathfrak{N}$ new peaks is p. This gives us that the probability of survival for infinite number of generations is positive, that is

$$\lim_{n\uparrow\infty} \mathrm{P}\left(\sum_{\mathfrak{i}\in\mathfrak{Z}_n} \mathfrak{X}_\mathfrak{i}\right) > 0.$$

Survival up to generation $\mathfrak{i} \in \mathfrak{Z}_k$ translates to very large peaks. We have that the overall solution reaches at least $\mathfrak{a} = \mathbf{L}^k$. We performed scaling such that an infinite number of generations fit in a bounded time interval. Generation $\mathfrak{i} \in \mathfrak{Z}_k$ lives only for $\mathbf{L}^{-\lambda k}$ amount of time on the original timescale, which corresponds to length one on the corresponding time scale for $u_{\mathfrak{i}}$. We have that

$$1 + \mathbf{L}^{-\lambda} + \mathbf{L}^{-2\lambda} + \cdots = \sum_{k=0}^{\infty} \mathbf{L}^{-k\lambda} = \frac{1}{1 - \mathbf{L}^{-\lambda}} =: \Xi < \infty, \tag{4.22}$$

for $\mathbf{L}$ large enough.

The previous fact of course gives us a contradiction with the assumption that $\mathfrak{t}(u) = \infty$ almost surely. What we proved so far is that large peaks will survive with positive probability. We have that $\lim_{N\to\infty} \mathrm{P}(\mathfrak{t}_N(u) \leq \Xi) > 0$, which proves Theorem 4.0.1. We also have that $\mathrm{P}(\mathfrak{t}(u) < \infty) > 0$.

4.3 General comments

This chapter and Theorem 4.0.1 is far less general than Theorem 3.1.1 in the previous chapter. We only considered a very narrow class of Lévy processes [α *stable and symmetric*] and a specific nonlinearity σ [$\sigma = |x|^\gamma$]. In every step of the proof in the present section, we relied heavily on the use of scaling. The reason is that under those specific conditions, '*everything*' scales well. Namely, it is the density of Lévy process, nonlinearity σ and the whole equation (SHE).

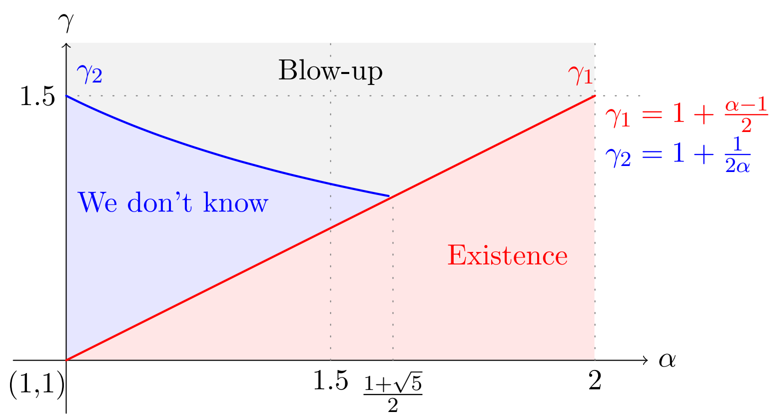

Figure 4.1. Phase diagram for a generator of a symmetric α stable Lévy process

APPENDIX A

INEQUALITIES

Theorem A.1 (Minkowski's integral inequality)**.** *[60, A.1, p. 271] Let F be a measurable function on $\mathcal{X} \times \mathcal{Y}$ with a product σ-algegra and σ-finite product measure $\mu \times \nu$, then*

$$\left(\int_{\mathcal{Y}} \left(\int_{\mathcal{X}} |F(x,y)|\, d\mu(x)\right)^p d\nu(y)\right)^{1/p} \leq \int_{\mathcal{X}} \left(\int_{\mathcal{Y}} |F(x,y)|^p\, d\nu\right)^{1/p} d\mu(x)$$

for any $1 \leq p < \infty$.

Theorem A.2 (Jensen's inequality)**.** *[16, p. 50] If φ is a convex function and $X, \varphi(X)$ are integrable random variables $(\mathrm{E}[X], \mathrm{E}[\varphi(X)] < \infty)$, then*

$$\varphi\left(\mathrm{E}[X]\right) \leq \mathrm{E}\left[\varphi\left(X\right)\right].$$

Theorem A.3 (Chebyshev's inequality)**.** *[16, p. 50] If φ is a scrictly postive and increasing function on $(0,\infty)$, $\varphi(u) = \varphi(-u)$, and X is a random variable such that $\mathrm{E}[\varphi(X)] < \infty$, then for each $u > 0$:*

$$\mathrm{P}\left(|X| \geq u\right) \leq \frac{\mathrm{E}\left[\varphi(u)\right]}{\varphi(u)}.$$

Theorem A.4 (Burkholder-Davis-Gundy Inequality)**.** *[11] Let $\{M_t\}_{t\geq 0}$ be a continuous L^2 martingale ($\forall T > 0, \sup_{t\leq T} \mathrm{E}\left[M_t^2\right] < \infty$). Then, for every $k \in [2,\infty)$,*

$$\|M_t\|^2_{L^k(\mathrm{P})} \leq 4k \left\|\langle M\rangle_t\right\|_{L^{k/2}(\mathrm{P})}.$$

Theorem A.5 (Hölder's inequality)**.** *[56, Thm. 3.5, p. 63] Let $p > 1$ and $1/p + 1/q = 1$. Let X be a measurable space with measure μ. Let f and g be measurable functions on X. Then*

$$\int_X |fg|\, d\mu \leq \left(\int_X |f|^p\, d\mu\right)^{1/p} \left(\int_X |g|^q\right)^{1/q}.$$

APPENDIX B

VARIOUS THEOREMS

Theorem B.1 (Kolmogorov's continuity theorem)**.** *[37, Thm. C.6, p. 107] Let $\{X_t\}_{t\in T}$ be a stochastic process, where $T \subset \mathbb{R}^m$ is measurable and bounded, and suppose that there exists finite real numbers $C > 0$ and $k > H$, such that*

$$\mathrm{E}\left[|X_t - X_s|^k\right]^{1/k} \leq C\varrho(t-s) \quad \textit{for all } s,t \in T.$$

Then X has a modification $\bar{X}$ that is Hölder continuous a.s. Moreover, $\bar{X}$ satisfies the following: For all $q \in (0, 1-(H/k))$ and $\delta \in (q, 1-(H/k)$, there exists a finite constant D, that does not depend on the numerical value of k, such that

$$\mathrm{E}\left[\sup_{s,t\in T, s\neq t}\left|\frac{\bar{X}_t - \bar{X}_s}{\varrho(t-s)^q}\right|^k\right] \leq (\delta-q)^{-1}\left(\frac{DC}{\delta}\right)^k k^{2H}\Lambda_T(2H-k+k\delta),$$

where

$$\Lambda_T(a) := \int_T dt \int_T ds \cdot \varrho(t-s)^{-a}.$$

Theorem B.2 (Kolmogorov's extension theorem)**.** *[51, I. Thm. 3.2, p. 32] Let E be a Polish space and $\mathscr{G}$ the σ-algebra of its Borel subsets, for any set T of indices and any projective family of probability measures on finite products, there exist a unique probability measure on $(E^T, \mathscr{G}^T)$ whose projections on finite products are the given family.*

Theorem B.3 (Kolmogorov's consistency/existence theorem)**.** *[22, Thm. 12.1.3] Let T by any set, m any function from T into $\mathbb{R}$ and C any function from $T\times T$ into $\mathbb{R}$ such that $C(s,t) = C(t,s)$ for all $s,t\in T$ and for any finite $F \subset T$, $\{C(s,t)\}_{s,t\in F}$ is a nonnegative definite matrix. Then there exists a Gaussian process $x_{tt\in T}$ with mean function m and covariance function C.*

Theorem B.4 (Doob-Meyer Decomposition)**.** *Let $X_t, t \geq 0$ be a continuous positive supermartingale of class DL. Then X can be decomposed as*

$$X_t = A_t + M_t$$

where A is a continuous nonincreasing process and M is a continuous martingale. Furthermore this decomposition is unique up to indistinguishability.

Proof. The proof of Doob-Meyer decomposition [38, 39] for a continuous case might be found in many different forms in literature [50, Chapter III], [53, VI.29] but not quite in the form we require. We will give a sketch of a proof based on [34, Chapter 1.4]. Theorem [34, Thm. 1.4.10] applied to submartingale $-X_t$ tells us that

$$X_t = A_t + M_t$$

where A_t is nonincreasing process and M_t is right-continuous martingale. This decomposition is also unique up to indistinguishability. Continuity of X_t gives us regularity [34, Def. 1.4.12] which in turn implies that A_t and M_t in the decomposition of X_t are both continuous by [34, Thm. 1.4.14]. □

Theorem B.5 (Unimodality for transition densities of α-stable process)**.** *[64, Thm. 2.7.4, p. 128] Transition densities $p_t(x), t > 0$ for α-stable symmetric Lévy process are unimodal with mode at 0. That is, for every $t > 0$, $p_t(x)$ as a function of x is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$.*

Theorem B.6 (Gronwall type inequality - Chindarov,1970)**.** *[2, Thm. 1.4, p. 5] Let $a(t)$,$b(t)$, $c(t)$, and $u(t)$ be continuous functions in $J = [\alpha, \beta]$, let $b(t)$ be nonnegative in J and suppose*

$$u(t) \le a(t) + \int_\alpha^t (b(s)u(s) + c(s))ds, \ t \in J.$$

Then

$$u(t) \le a(t) + \int_\alpha^t (a(s)b(s) + c(s)) \exp\left(\int_s^t b(\tau)d\tau\right) ds.$$

Theorem B.7 (Tonelli's theorem)**.** *[56, Thm. 8.8(a), p. 164] Let $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ be σ-finite measure space and let f be a nonnegative $(\mathscr{F} \times \mathscr{G})$-measurable function on $X \times Y$, then*

$$\int_X \left(\int_Y f(x,y)d\nu\right) d\mu = \int_{X\times Y} f(x,y)d(\mu \times \nu) = \int_Y \left(\int_X f(x,y)d\mu\right) d\nu.$$

Theorem B.8 (Itô's formula)**.** *[51, IV. Thm. 3.3, p. 138] Let $X = (X^1, \ldots, X^d)$ be a continuous vector semimartingale and $F \in C^2(\mathbb{R}^d, \mathbb{R})$; then, $F(X)$ is a continuous semimartingale and*

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)dX_s^i + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)d\langle X^i, X^j\rangle_s$$

Theorem B.9 (Reflection principle for Brownian motion)**.** *[51, Prep. 3.7, p. 100] Let B_t be a standard Brownian motion and let*

$$S_t := \sup_{s \le t} B_s,$$
$$T_a := \inf\{t \ge 0 : S_t \ge a\},$$

then for every $a > 0$ and $t \ge 0$,

$$\mathrm{P}\left(S_t \ge a\right) = \mathrm{P}\left(T_a \le t\right) = 2\,\mathrm{P}\left(B_t \ge a\right) = \mathrm{P}\left(|B_t| \ge a\right).$$

Theorem B.10 (Knight's Theorem)**.** *[34, Thm. 3.4.13, p. 179] Let $M_t = (M_t^{(1)}, \dots, M_t^{(d})$ be a continuous $\mathscr{F}_t$ adapted local martingale, such that $\lim_{t\to\infty} M^{(i)} > K > 0$ almost surely, and*

$$\langle M^{(i)}, M^{(j)} \rangle_t = 0;\ 1 \le i \ne j \le d,\ 0 \le t < \infty.$$

Define

$$T_i(s) = \inf\{t \ge 0; \langle M^{(i)} \rangle_t \ge s\}; 0 \le s < K, 1 \le i \le d,$$

so that for each i and s, the random time $T_i(s)$ is a stopping for the filtration $\mathscr{F}_t$. Then the processes

$$B_s^{(i)} := M_{T_i(s)}^{(i)};\ 0 \le s < K,\ 1 \le i \le d,$$

are independent, standard one-dimensional Brownian motions.

Theorem B.11 (Stochastic Fubini)**.** *[37, Thm. 5.10] Suppose that $\Phi \in \mathscr{P}_M$ [see def. of $\mathscr{P}_M$ on p. 14], $p_t(x) = 1/\sqrt{2\pi t}\exp(-x^2/2t)$, $\varphi \in \mathscr{S}$ and η is a space-time white noise. Then, almost surely:*

$$\int_{\mathbb{R}} \varphi(x) \left(\int_0^T p_{t-s}(x-y)\Phi(s,y)\eta(ds,dy)dx \right) = \int_0^T \int_{\mathbb{R}} \left(p_{t-s} * \varphi(y)\right) \Phi(s,y)\eta(ds,dy).$$

APPENDIX C

Index

REFERENCES

[1] P. W. Anderson, *Absence of diffusion in certain random lattices*, Physical Review, 109 (1958), pp. 1492–1505.

[2] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of Mathematics and its Applications (East European Series), Kluwer Academic Publishers, 1992.

[3] R. M. Balan and D. Conus, *Intermittency for the wave and heat equations with fractional noise in time*, To be published in the Annals of Probability.

[4] R. M. Balan and C. A. Tudor, *Stochastic heat equation with multiplicative fractional-colored noise*, Journal of Theoretical Probability, 23 (2010), pp. 834–870.

[5] J. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, The Quarterly Journal of Mathematics, 28 (1977), pp. 473–486.

[6] L. Bertini and N. Cancrini, *The stochastic heat equation: Feynman-kac formula and intermittence*, Journal of Statistical Physics, 78 (1995), pp. 1377–1401.

[7] L. Bertini, N. Cancrini, and G. Jona-Lasinio, *The stochastic burgers equation*, Communications in Mathematical Physics, 165 (1994), pp. 211–232.

[8] P. Bezdek, *On weak convergence of stochastic heat equation with colored noise*, To be published in the Stochastic Processes and their Applications.

[9] P. Billingsley, *Weak Convergence of Measures: Applications in Probability*, CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1971.

[10] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, 1999.

[11] E. Carlen and P. Kree, *l^p estimates on iterated stochastic integrals*, The Annals of Probability, 19 (1991), pp. 354–368.

[12] L. Chen and R. Dalang, *Moments, intermittency and growth indices for the nonlinear fractional stochastic heat equation*, Stochastic Partial Differential Equations: Analysis and Computations, 3 (2015), pp. 360–397.

[13] L. Chen and R. C. Dalang, *Moments, intermittency and growth indices for the nonlinear fractional stochastic heat equation*, Stochastic Partial Differential Equations: Analysis and Computations, 3 (2015), pp. 360–397.

[14] L. Chen and K. Kim, *On comparison principle and strict positivity of solutions to the nonlinear stochastic fractional heat equations*, To be published in the Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.

[15] P.-L. Chow, *Explosive solutions of stochastic reactiondiffusion equations in mean L^p-norm*, Journal of Differential Equations, 250 (2011), pp. 2567 – 2580.

[16] K. Chung, *A Course in Probability Theory*, Academic Press, 2001.

[17] D. Conus, M. Joseph, D. Khoshnevisan, and S.-Y. Shiu, *On the chaotic character of the stochastic heat equation, ii*, Probability Theory and Related Fields, 156 (2013), pp. 483–533.

[18] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2014.

[19] R. Dalang, *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s*, Electronic Journal of Probability, 4 (1999), pp. no. 6, 1–29.

[20] R. Dalang, D. Khoshnevisan, and F. Rassoul-Agha, *A Minicourse on Stochastic Partial Differential Equations*, no. 1962 in A Minicourse on Stochastic Partial Differential Equations, Springer, 2009.

[21] L. Debbi and M. Dozzi, *On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension*, Stochastic Processes and their Applications, 115 (2005), pp. 1764 – 1781.

[22] R. Dudley, *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2002.

[23] W. Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Annals of Mathematics, 55 (1952), pp. 468–519.

[24] M. Foondun and D. Khoshnevisan, *Intermittence and nonlinear parabolic stochastic partial differential equations*, Electronic Journal of Probability, 14 (2009), pp. no. 21, 548–568.

[25] M. Foondun and D. Khoshnevisan, *On the stochastic heat equation with spatially-colored random forcing*, Transactions of the American Mathematical Society, 365 (2013), pp. 409–458.

[26] H. Fujita, *On the blowing up of solutions of the cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, Faculty of Science, The University of Tokyo, (1966), pp. 105–113.

[27] V. A. Galaktionov and J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete and Continuous Dynamical Systems, 8 (2002), pp. 399–434.

[28] I. Gelfand and G. Chilov, *Generalized Functions: Properties and Operations*, Generalized functions, Academic Press, 1964.

[29] I. M. Gelfand and N. Y. Vilenkin, *Generalized Functions: Vol.: 4. : Applications of Harmonic Analysis*, Academic Press, 1964.

[30] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 2007.

[31] M. Hairer, *Solving the KPZ equation.*, Annals of Mathematics. Second Series, 178 (2013), pp. 559–664.

[32] M. Hairer and J. Voss, *Approximations to the stochastic burgers equation*, Journal of Nonlinear Science, 21 (2011), pp. 897–920.

[33] Y. Hu and D. Nualart, *Stochastic heat equation driven by fractional noise and local time*, Probability Theory and Related Fields, 143 (2008), pp. 285–328.

[34] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, Springer New York, 2014.

[35] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic scaling of growing interfaces*, Physical Review Letters, 56 (1986), p. 889.

[36] D. Khoshnevisan, *Probability*, Graduate Studies in Mathematics, American Mathematical Society, 2007.

[37] ——, *Analysis of Stochastic Partial Differential Equations*, vol. 119, American Mathematical Society, 2014.

[38] P. A. Meyer, *A decomposition theorem for supermartingales*, Illinois Journal of Mathematics, 6 (1962), pp. 193–205.

[39] ——, *Decomposition of supermartingales: The uniqueness theorem*, Illinois Journal of Mathematics, 7 (1963), pp. 1–17.

[40] C. Mueller, *Long time existence for the heat equation with a noise term*, Probability Theory and Related Fields, 90 (1991), pp. 505–517.

[41] C. Mueller, *On the support of solutions to the heat equation with noise*, Stochastics and Stochastic Reports, 37 (1991), pp. 225–245.

[42] C. Mueller, *The critical parameter for the heat equation with a noise term to blow up in finite time*, Annals of Probability, (2000), pp. 1735–1746.

[43] C. Mueller and R. Sowers, *Blowup for the heat equation with a noise term*, Probability Theory and Related Fields, 97 (1993), pp. 287–320.

[44] C. Mueller and R. Tribe, *A singular parabolic anderson model*, Electronic Journal of Probability, 9 (2004), pp. no. 5, 98–144.

[45] D. Nualart, *The Malliavin Calculus and Related Topics*, Probability and Its Applications, Springer Berlin Heidelberg, 2006.

[46] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Hochschultext / Universitext, Springer, 2003.

[47] W. F. Osgood, *Beweis der existenz einer lösung der differentialgleichung* $dy/dx = f(x,y)$ *ohne hinzunahme der cauchy-lipschitz'schen bedingung*, Monatshefte für Mathematik und Physik, 9, pp. 331–345.

[48] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Springer New York, 1992.

[49] G. Prato, A. Debussche, and R. Temam, *Stochastic burgers' equation*, Nonlinear Differential Equations and Applications NoDEA, 1, pp. 389–402.

[50] P. Protter, *Stochastic Integration and Differential Equations*, Stochastic Modelling and Applied Probability, Springer Berlin Heidelberg, 2013.

[51] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, vol. 293, Springer Berlin Heidelberg, 1999.

[52] T. Rippl and A. Sturm, *New results on pathwise uniqueness for the heat equation with colored noise*, Electronic Journal of Probability, 18 (2013), pp. no. 77, 1–46.

[53] L. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales: Volume 2, Itô Calculus*, Cambridge Mathematical Library, Cambridge University Press, 2000.

[54] H. Royden and P. Fitzpatrick, *Real Analysis*, Featured Titles for Real Analysis Series, Prentice Hall, 2010.

[55] W. Rudin, *Principles of Mathematical Analysis*, International series in pure and applied mathematics, McGraw-Hill, 1976.

[56] ——, *Real and Complex Analysis*, Mathematics series, McGraw-Hill, 1987.

[57] M. Sanz-Solé and M. Sarrà, *Path properties of a class of gaussian processes with applications to spdes*, Canadian Mathematical Society, Conference Proceedings, 28 (2000), pp. 303–316.

[58] M. Sanz-Solé and M. Sarrà, *Hölder continuity for the stochastic heat equation with spatially correlated noise*, Seminar on Stochastic Analysis, Random Fields and Applications III, (2002), pp. 259–268.

[59] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[60] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Monographs in harmonic analysis, Princeton University Press, 1970.

[61] J. B. Walsh, *An introduction to stochastic partial differential equations*, in École d'Été de Probabilités de Saint Flour XIV - 1984, vol. 1180 of Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1986, pp. 265–439.

[62] H. W. Watson and F. Galton, *On the probability of the extinction of families*, The Journal of the Anthropological Institute of Great Britain and Ireland, 4 (1875), pp. 138–144.

[63] M. J. Wichura, *Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters*, The Annals of Mathematical Statistics, 40 (1969), pp. 681–687.

[64] V. Zolotarev, *One-dimensional Stable Distributions*, Translations of mathematical monographs, American Mathematical Society, 1986.